VI Nonlinear Dynamical Systems: Worked Examples, Perspectives and Open Problems.

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1. Basic terminology and an outline of the program

1.1. Fundamental concepts of dynamical systems theory

A first and important step in the construction of a mathematical model of a (biological) system consists of the choice of a state space. The mathematical *state* should be a convenient representation of those physiological, chemical, physical and other relevant properties which in our conception of (or hypotheses about) reality uniquely determine the future, in the sense that for a given time course of experimental or environmental conditions (the input) we obtain a unique time course of the quantities we are interested in (the output). In section III.2 we made some remarks about the choice of a state space and about the construction of a state space from input-output data. Moreover, we presented a precise mathematical reformulation of the intuitive definition of "state" above in terms of a family of operators with a semigroup property.

When we restrict ourselves to time-independent inputs, things are a little simpler than in the general case. The input then is just a fixed parameter which we may suppress in the notation. In Chapter II we have treated some aspects of the theory of such autonomous (i.e., time-translation invariant) dynamical systems under the additional assumption of *linearity* and in the present chapter we shall discuss some aspects of autonomous *nonlinear* dynamical systems. Concerning outputs we confine ourselves to the remark that one may choose any (continuous) function of the state as output mapping.

We begin by introducing some terminology. Although certain states (for instance negative weight) might lack any biological interpretation, one can always define the dynamics on the whole linear state space if the system is linear, simply by using the linearity. This, of course, is no longer possible if the system is nonlinear. So let X be a Banach space and C a closed subset of X. A (semi) *dynamical system* on C is a mapping $u:\mathbb{R}^+ \times C \to C$ such that

(i) $u(\cdot,\phi):\mathbb{R}^+ \to C$ is continuous from above in $t=0$)

(ii) $u(t,\cdot):C \to C$ is continuous (for any $t \geqslant 0$)

(iii) $u(0,\phi) = \phi$

(iv) $u(t+s,\phi) = u(t,u(s,\phi)),\ \forall t,s \in \mathbb{R}^+,\ \phi \in C$.

REMARK 1.1.1. One can show that these properties imply that $u:\mathbb{R}^+ \times C \to C$ is continuous, *i.e.* u is continuous as a function of two variables.

A (one-parameter) *strongly continuous semigroup* on C is a family $\{T(t)\}_{t \geqslant 0}$ of (not necessarily linear) mappings from C into C such that

(i) $t \mapsto T(t)\phi$ is continuous from $\mathbb{R}^+$ to C

(ii) for each $t \geqslant 0$, $T(t):C \to C$ is continuous

(iii) $T(0) = I$ (where I denotes the identity operator on C)

(iv) $T(t+s) = T(t)T(s),\ \forall t,s \in \mathbb{R}^+$.

The *infinitesimal generator* of $T(t)$ is the operator

$$A\phi = \lim_{t \downarrow 0} \frac{1}{t}(T(t)\phi - \phi)$$

for all $\phi \in \mathfrak{D}(A)$, the subset of C for which this limit exists.

The above notions of a dynamical system and a strongly continuous semigroup are equivalent in the sense that for a given closed subset C the identification

$$u(t,\phi) = T(t)\phi$$

gives a one-to-one correspondence between all dynamical systems on the one and all semigroups on the other hand. Therefore we can use these names interchangeably without causing confusion.

The outcome of the model building phase usually is a rather loosely defined "generator", and a first mathematical task is to *prove* that indeed one can associate a dynamical system with the model. Some results analogous to the Hille-Yosida Theorem II.2.6 exist for the nonlinear case (see PAZY, 1983b, for a nice survey), but as in the linear case one can usually follow a more direct and easy road. We shall demonstrate this road in section 2 by means of an example from mathematical epidemiology. The basic idea is to use integration along characteristics to convert the formal initial value problem into an integral equation and to use a so-called contraction mapping argument to prove existence and uniqueness of solutions of the integral equation on an appropriately small time interval. By repetition of the argument one can continue the solution as long as it remains bounded. Next one has to derive *a priori estimates*, guaranteeing that the solution cannot blow up in finite time, in order to obtain a solution defined for $0 \leq t < \infty$. Here the choice of C matters (of course C should also be adjusted to the biological interpretation; quite often, however, the definition of C involves only a precise mathematical formulation of obvious biological constraints; see section 2 for an example). Finally the circle may be closed by computing the infinitesimal generator of the semigroup. This amounts to a precise redefinition of the original "generator" and it provides us with a precise notion of "solution" (see Chapter II for a detailed example in the linear case).

After the existence and uniqueness of solutions to the initial value problem have been demonstrated, a more difficult task begins: one has to draw conclusions about biologically relevant aspects of the behaviour of the dynamical system. In the linear case we used spectral theory (Chapter II), Laplace transforms (Chapter IV) or both (Chapter V). Neither of the two has an analogue in the nonlinear case.

The behaviour of a nonlinear dynamical system can be essentially richer and more complicated than the behaviour of a linear dynamical system. In order to describe and classify various possibilities we need a lot more terminology, some of which we present below while referring to GUCKENHEIMER & HOLMES (1983), WALKER (1980) and SAPERSTONE (1981), for an introduction to the relevant literature.

Given the *initial state* ϕ at time zero, $T(t)\phi$ is the *state* at time t. The set

$$\Gamma^+(\phi) = \{T(t)\phi \mid t \geq 0\}$$

is called the *orbit* * starting at ϕ. Some rather simple orbits are those consisting of just one point or a closed curve of points. If

$$T(t)\hat{\phi} = \hat{\phi} \quad , \text{ for all } t \geq 0 \ ,$$

then $\hat{\phi}$ is called an *equilibrium* or a *steady state.* If

$$T(\tau)\phi_p = \phi_p \quad , \text{ for some } \tau > 0 \ ,$$

we call ϕ_p a *periodic point* with period τ and $\Gamma^+(\phi_p) = \{T(t)\phi_p | 0 \leq t < \tau\}$ is called a *periodic orbit.* The minimal $\tau > 0$ with the property above is called the minimal period, but usually the adjective "minimal" will be omitted.

We will be especially interested in those orbits or more complicated sets, which in some sense attract (or at least do not repel) orbits starting nearby. The distance dist (ϕ, V) between a point $\phi \in X$ and a non-empty subset V of X is defined as

$$dist(\phi, V) = \inf\{\|\phi - \psi\| \mid \psi \in V\} \, .$$

With respect to $\{T(t)\}$ we call a subset V of X

i) *stable* if $\forall \epsilon > 0 \exists \delta = \delta(\epsilon) > 0$ such that $\forall \psi \in C$ with $\text{dist}(\psi, V) < \delta$ necessarily dist $(T(t)\psi, V) < \epsilon$, $\forall t \geq 0$;

ii) *unstable* if V is *not* stable;

iii) *asymptotically stable* if V is stable and in addition $\exists \delta > 0$ such that $\forall \psi \in C$ with dist $(\psi, V) < \delta$ necessarily $\lim_{t \to \infty}$ dist $(T(t)\psi, V) = 0$.

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* An orbit is sometimes also called a *trajectory,* but perhaps it is better to reserve the name trajectory for the subset $\{(t, T(t)\phi) | t \geq 0\}$ of $\mathbb{R} \times C$.

iv) *exponentially stable* if $\exists\delta>0$ such that $\exists\alpha = \alpha(\delta)>0$ and $\exists M = M(\delta)>0$ such that $\forall\psi\in C$ with dist $(\psi, V)<\delta$ necessarily dist $(T(t)\psi, V)\leq Me^{-\alpha t}$ dist (ψ, V), $\forall t\geq 0$.

If in the last two situations one may choose δ arbitrarily large one adds the adverb "*globally*". If V is an orbit itself one speaks about *orbital stability* etc.. If an asymptotically stable set V consists of more than one point (for instance, V might be a periodic orbit), one says that orbits $\Gamma^+(\psi)$ converging to V have an *asymptotic phase* if $\phi = \phi(\psi)\in V$ exists such that $\lim_{t\to\infty}(T(t)\psi - T(t)\phi) = 0$. The *domain of attraction* of an asymptotically stable set V is the set $\{\psi\in C \mid \lim_{t\to\infty} \text{dist}\,(T(t)\psi, V) = 0\}$.

As a side-remark we mention that other stability concepts exist. The above definitions correspond to so-called *Lyapunov-stability*.

Not all stable sets are interesting. For instance, if $\hat{\phi}$ is a globally exponentially stable steady state, *any* set containing $\hat{\phi}$ is stable. We want to concentrate on those sets which are mapped into themselves by the semigroup and which, moreover, are related to the asymptotic (large time) behaviour of orbits. A subset M of C is called *positively invariant* if $T(t)\phi\in M$ for all $\phi\in M$ and all $t\geq 0$. We say that a point $\phi\in C$ has a *backward extension* if there exists a mapping $F:\mathbb{R}\times C\to C$ such that $F(0,\phi) = \phi$ and $F(t+s,\phi) = T(t)F(s,\phi)$ for all $s\in\mathbb{R}$ and $t\in\mathbb{R}_+$ (note that in the "forward" time direction F simply describes the action of $T(t)$ and that, at least in principle, several such backward extensions might exist since the semigroup operators might not be one-to-one; see *e.g.* section II.11). A subset M of C is called *invariant* if it is positively invariant and in addition each point of M has a backward extension which belongs to M.

EXERCISE 1.1.2. Show that (a) if $\hat{\phi}$ is an equilibrium then $\{\hat{\phi}\}$ is invariant, (b) if ϕ_p is a periodic point then $\Gamma^+(\phi_p)$ is invariant, (c) the union of two invariant sets is invariant, (d) a subset M of C is invariant if and only if $T(t)M = M$ for all $t\geq 0$.

For any $\phi\in C$ we define

$$\omega(\phi) = \{\psi\in C \mid \exists\{t_k\} \text{ such that } t_k\to\infty \text{ and } T(t_k)\phi\to\psi \text{ for } k\to\infty\}$$

and call it the *ω- limit set* of ϕ.

EXERCISE 1.1.3. Show that $\omega(\phi)$ is closed and positively invariant.

A much stronger result holds if the orbit $\Gamma^+(\phi)$ is precompact (i.e. the closure $\overline{\Gamma^+(\phi)}$ is compact). For the proof see, for instance, Walker (1980), page 167.

THEOREM 1.1.4. *If* $\Gamma^+(\phi)$ *is precompact, then* $\omega(\phi)$ *is nonempty, compact, connected and invariant. Moreover* $\lim_{t\to\infty}$ dist $(T(t)\phi, \omega(\phi)) = 0$.

REMARK 1.1.5. In the description of the orbit structure of dynamical systems it has proved useful to pay special attention to points which not necessarily belong to any ω-limit set, but nevertheless show some form of recurrent behaviour. This has led to the notions of nonwandering and chain recurrent points. See Guckenheimer & Holmes (1983).

The ideal second step in the analysis of a structured population model would be to trace all ω-limit sets and to determine their stability properties and their domain of attraction. As a rule it is impossible to obtain such a complete global overview of the orbit structure, but in special cases one of the following two methods might help to attain the end:

Lyapunov functions and the Invariance Principle.

Let $V:C\to\mathbb{R}$ be a continuous function. The derivative of V along the orbit starting in ϕ at $t = 0$ is given by

$$\lim_{t\downarrow 0} \frac{1}{t}\,(V(T(t)\phi) - V(\phi)),$$

but this limit might not exist. Therefore we define

$$\dot{V}(\phi) = \liminf_{t\downarrow 0} \frac{1}{t}\,(V(T(t)\phi) - V(\phi)), \tag{1.1}$$

where we allow that $\dot{V}(\phi) = -\infty$. The function V is called a *Lyapunov function* if $\dot{V}(\phi)\leq 0$ for all $\phi\in C$. So let V be a

Lyapunov function. We define

$$E = \{\phi \in C \mid \dot{V}(\phi) = 0\}$$

and

$$\mathcal{S} = \text{largest invariant subset of } E\,.$$

THEOREM 1.1.6. (Invariance Principle). *Let V be a Lyapunov function for $T(t)$ and assume that $\Gamma^+(\phi)$ is precompact. Then*

$$\lim_{t \to \infty} \operatorname{dist}(T(t)\phi, \mathcal{S}) = 0\,.$$

REMARKS 1.1.7. (i) We did not state the most general formulation of the Invariance Principle. See Walker (1980), section IV.4 or LaSalle (1976).
(ii) Theorem 1.1.6. is particularly useful if one can show that $\mathcal{S}$ is a discrete set of points. In section 4 we shall illustrate the use of the Invariance Principle in the simplest case in which $\mathcal{S}$ is just a singleton.
(iii) The difficulty with applying this strong and beautiful result is, of course, to find a Lyapunov function for the concrete problem at hand. Quite often this turns out to be an impossible task.

Monotonicity methods

Let X_+ be a closed convex cone in X (see Chapter V for relevant definitions, in particular of the $\leqslant$ symbol below). Assume that C is a subset of X_+. The semigroup $T(t)$ is called *monotone* if $\phi \leqslant \psi$ implies that $T(t)\phi \leqslant T(t)\psi$ for all $t \geqslant 0$. (So in the linear case monotonicity is nothing but positivity.) One can prove the following result.

THEOREM 1.1.8. *Suppose that*

(i) *$T(t)$ is a monotone semigroup*

(ii) *there exists precisely one equilibrium $\hat{\phi} \in C \setminus \{0\}$*

(iii) *for every $\phi \in C$, $\phi \neq 0$, there exist $t_0 > 0$ and $\underline{\phi}$, $\bar{\phi} \in C$ such that $0 < \underline{\phi} \leqslant T(t_0)\phi \leqslant \bar{\phi}$ and, moreover, $T(t)\underline{\phi}$ is increasing and $T(t)\bar{\phi}$ is decreasing with respect to t*

(iv) *orbits are precompact*

then $\lim_{t \to \infty} T(t)\phi = \hat{\phi}$ *for all* $\phi > 0$.

EXERCISE 1.1.9. Prove the above statement.

So under rather strong monotonicity assumptions the dynamics become rather simple. In section 4 we shall illustrate this general idea by means of an example from cell kinetics. Also see the contribution of Thieme to part B. We refer to Hirsch (1984a,b), Matano (1984) and Matano & Hirsch (in preparation) for a number of interesting results which hold without conditions like (ii) and (iii) above.

In general neither of these two methods will be applicable and one has to resort to a less ambitious program:

i) Trace all equilibria and determine their (local) stability.

ii) Find criteria for the existence and the (local) stability of periodic orbits.

iii) Find out whether there exist more complicated attracting invariant sets like tori or even so-called strange attractors with a Cantor set structure.

The last two items are fairly complicated already and computer experiments are usually an indispensable tool for such investigations.

It is frequently convenient to concentrate on qualitative changes of the orbit structure which may or may not occur when model parameters are varied. This method, which goes under the heading of *bifurcation theory*, is the mathematical counterpart of the experimental technique to study a system by slowly changing one of its controllable parameters while observing what happens.

In the next subsection we present some general principles from *local* stability and bifurcation theory in the context of ordinary differential equations, since for those the results are well-established and rigorously proved. In subsection

1.3 we speculate about some *global* aspects and about the practical meaning of stability. Then follow a number of examples from structured population dynamics which are intended to equip the reader with some feeling for the subject, and to illustrate various mathematical methods and techniques. All examples have in common that they can be reduced to an integral or an integro-differential equation, and as such are rather special. In a final section we mention several open problems for a more general class of first order functional partial differential equations. Most of these are concerned with generalizing the local stability and bifurcation results of subsection 1.2 to this class of equations. Finally we refer to Hale, Magalhães & Oliva (1984) for an introduction to the global and generic theory of infinite dimensional dynamical systems, while noting that, as far as we know, no work in this spirit has been done for the kind of functional partial differential equations corresponding to structured population models.

So all together we try to present our wishful thinking about a comprehensive mathematical theory of *nonlinear* structured population models in the hope that our readers feel stimulated to contribute to its creation.

1.2. Linearized stability and bifurcation theory in the context of ordinary differential equations

In this subsection we shall sketch some of the basic results of nonlinear analysis without giving any proof. We refer to such sources as Hartman (1964), Hale (1969), Hirsch & Smale (1974), Guckenheimer & Holmes (1983), Iooss & Joseph (1980) and Chow & Hale (1982) for a detailed account of the much more comprehensive theory. Here our aim is just to help the mathematically uninitiated reader to build up some knowledge of the terminology and some intuition for the main results and problems.

In this subsection f denotes a function from $\mathbb{R}^N$ into $\mathbb{R}^N$ which is at least C^1-smooth and, moreover, is such that $f(0) = 0$. Hence $x = 0$ is an equilibrium for the ordinary differential equation

$$\frac{dx}{dt} = f(x) \tag{1.2.1}$$

or, in the terminology of the last subsection, for the dynamical system $x(t,x_0)$, where $x(t,x_0)$ denotes the solution of (1.2.1) with initial condition $x(0,x_0) = x_0$. In this context the function f is called a *vector field* on $\mathbb{R}^N$.

We shall concentrate on the behaviour of orbits in a neighbourhood of $x = 0$. Let $Df(0)$ denote the linearization of f at $x = 0$ (i.e., the Jacobi matrix of partial derivatives). An obvious question is: can one determine the stability of $x = 0$ from the stability of $x = 0$ with respect to the linear semigroup $e^{Df(0)t}$? Since the latter is completely determined by the eigenvalues of $Df(0)$ one can equivalently ask whether these have a decisive influence on the stability of $x = 0$. The positive answer is called the *Principle of Linearized Stability:*

THEOREM 1.2.1. (Poincaré-Lyapunov)

i) *If all eigenvalues of $Df(0)$ have negative real part, $x = 0$ is exponentially stable*

ii) *If at least one of the eigenvalues of $Df(0)$ has positive real part, $x = 0$ is unstable.*

If $Re\lambda \leqslant 0$ for all eigenvalues λ, with equality for at least one λ, the higher order terms come out of their subordinate position and control the situation.

The local equivalence between the semigroup and its linearization is described by the much stronger

THEOREM 1.2.2. (Hartman-Grobman) *If none of the eigenvalues of $Df(0)$ lies on the imaginary axis* * *then there exists a homeomorphism* † *h defined on a neighbourhood U of $x = 0$ which maps (restrictions to U of) orbits of the nonlinear semigroup onto orbits of the linear semigroup $e^{Df(0)t}$, preserving the direction in which they are traversed in the course of time.*

Let $S(t)$ denote the nonlinear semigroup corresponding to (1.2.1) and define the *local stable and unstable manifolds* by

$$W^s_{loc}(0) = \{x \in U \mid S(t)x \to 0 \text{ for } t \to \infty \text{ and } S(t)x \in U,\ \forall t \geqslant 0\} \tag{1.2.2}$$

$$W^u_{loc}(0) = \{x \in U \mid S(t)x \to 0 \text{ for } t \to -\infty \text{ and } S(t)x \in U,\ \forall t \leqslant 0\} \tag{1.2.3}$$

then we have

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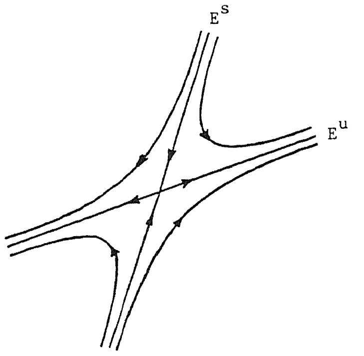
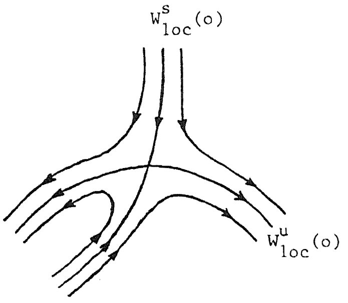
* An equilibrium with this property is called *hyperbolic.*

† A homeomorphism is simply a continuous map, having a continuous inverse.

Theorem 1.2.3. (the saddle point property) *If none of the eigenvalues of* $Df(0)$ *lies on the imaginary axis there exist local stable and unstable manifolds* $W^s_{loc}(0)$ *and* $W^u_{loc}(0)$ *of the same dimensions as the invariant generalized eigenspaces* E^s *and* E^u *of the linearized semigroup* $e^{Df(0)t}$ *corresponding to, respectively, the eigenvalues with negative real part and those with positive real part. The manifolds* $W^s_{loc}(0)$ *and* $W^u_{loc}(0)$ *are tangent to, respectively,* E^s *and* E^u *in* $x = 0$ *and they are as smooth as the function f.*



linear



non linear

Homeomorphisms as in Theorem 1.2.2 generate a local equivalence relation among semigroups, which obviously can be extended to a global equivalence relation by omitting the special emphasis on a neighbourhood of an equilibrium point. A function f is called *structurally stable* if all functions in a C^1-neighbourhood of f belong to the same equivalence class. This notion might seem rather abstract at first sight, but actually it is suggested by the following practical considerations. If we model some real world phenomenon in terms of differential equations we always make idealizations and approximations while, moreover, leaving many minor effects completely out of consideration. So at best the real dynamics is close to the dynamics described by the differential equations. If the latter is "robust" (proof against small perturbations of the model assumptions or, in other words, of the differential equations themselves) we may trust its predictions, but otherwise we have to reexamine our modelling assumptions quite carefully.

In numerous applications f will naturally depend on *parameters* in such a way that the equivalence class changes when these parameters are varied (or, in other words, the parametrized family of vector fields cuts the "boundary" between two (or more) equivalence classes). As far as *local* equivalence is concerned, Theorem 1.2.2 tells us that this happens exactly when some eigenvalues of $Df(0)$ cross the imaginary axis for some specific value of the parameters. So if eigenvalues move from one halfplane into the other as parameters are varied, we expect to see qualitative changes in the (local) orbit structure. *Bifurcation theory* embraces the *classification* of possible qualitative changes and the development of constructive *algorithms* to determine the actual change that will occur in some concrete situation. In the simplest cases, viz.,

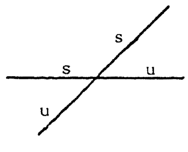
(i) a simple real eigenvalue crosses the imaginary axis with positive speed

(ii) a pair of complex conjugate simple eigenvalues crosses the imaginary axis with positive speed

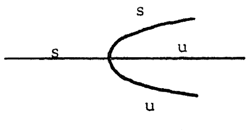
everything is thoroughly understood.

Example 1.2.4. Let x denote the (one-dimensional) state variable and μ the (one-dimensional) parameter. We draw the zero set of $f = f(x,\mu)$ in the (x,μ)-plane for various functions f (see below).

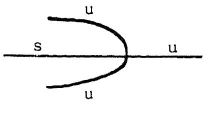
Such pictures are called *bifurcation diagrams*. In (i)-(iv) the curve $x = 0$ is called the *trivial branch* and the other curves are called the *bifurcating branches*. The word *"bifurcation point"* refers to $\mu = 0$ in some texts and to $(x,\mu) = (0,0)$ in others.



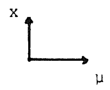
(i) $f = x(\mu - x)$
transcritical

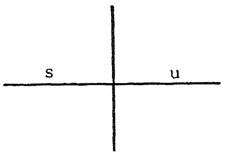


(ii) $f = x(\mu - x^2)$
pitchfork

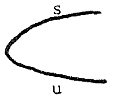


(iii) $f = x(\mu + x^2)$
pitchfork

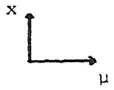




(iv) f = xμ
vertical



(v) f = $\mu - x^2$
saddle-node

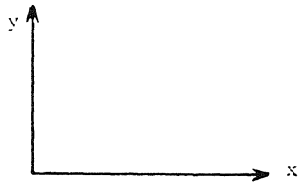


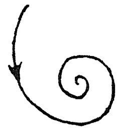
Bifurcation theory tells us that the simple one-dimensional examples above are, locally near the bifurcation point, representative for the general case of a one-dimensional parameter μ and a higher (perhaps infinite) dimensional state variable x, under the stated condition (i) on the eigenvalue. Moreover, it provides us with a *constructive procedure* (based on a combination of the so-called Lyapunov-Schmidt reduction technique and the contraction mapping principle or, if you wish, the closely related implicit function theorem; an alternative approach uses reduction to normal forms; see Guckenheimer & Holmes, 1983) to calculate approximations to the bifurcating branches which cannot, of course, be calculated explicitly in general. Thus one can decide which of the pictures of Example 1.2.4 covers the situation at hand

EXAMPLE 1.2.5. In terms of polar coordinates $x = r\cos\theta$, $y = r\sin\theta$ the system

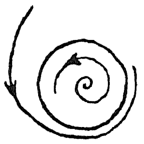
$$\dot{x} = \mu x - \omega y \pm x(x^2 + y^2)$$
$$\dot{y} = \omega x + \mu y \pm y(x^2 + y^2)$$

reduces to $\dot{r} = r(\mu \pm r^2)$, $\dot{\theta} = \omega$. So with the $-$ sign we have



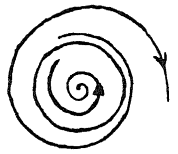


$\mu < 0$

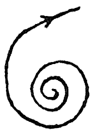


$\mu > 0$

whereas with the + sign we have

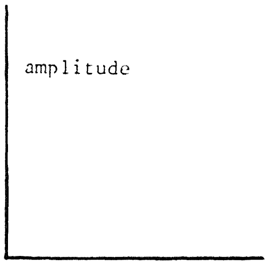


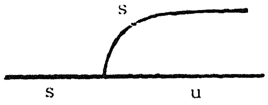
$\mu < 0$



$\mu > 0$

In other "pictures"





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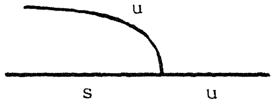

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Again this simple example is representative for the general situation covered by condition (ii) above. The origination of a *periodic solution* from an equilibrium (which changes its stability type due to a pair of complex eigenvalues crossing the imaginary axis) is called a *Hopf bifurcation.*

EXERCISE 1.2.6. Check that the stability of the equilibria of $\dot{x} = f(x,\mu)$ with f as in $(i)-(v)$ in Example 1.2.4. is as indicated in the diagrams. Similarly determine the stability of the periodic solutions in Example 1.2.5.

If the trivial solution is stable for $\mu<0$ and unstable for $\mu>0$, then the bifurcating solutions which exist for $\mu>0$ are called *supercritical* and those which exist for $\mu<0$ are called *subcritical.* The outcome of Exercise 1.2.6 illustrates the general rule

supercritical bifurcating solutions are stable and subcritical bifurcating solutions are unstable

which is called the *principle of exchange of stability.*

The situation is essentially more complicated if eigenvalues have higher multiplicity, several eigenvalues cross simultaneously, eigenvalues hesitate or change their mind while lying on the imaginary axis (i.e. the derivative of the real part with respect to the parameter vanishes). The study of such situations is in full progress today and many results are known (CHOW & HALE, 1982, GUCKENHEIMER & HOLMES, 1983). Especially in the case of multiple eigenvalues or several (pair of) eigenvalues it is "natural" to consider families with more than one parameter in order to produce, as it is called in expressive language, a universal or partial unfolding of the singularity. Quite often it turns out that the local picture thus obtained gives, in fact, global information (see GOLUBITSKY and SCHAEFFER (1984) for a discussion of this phenomenon and the related concept of "organizing center").

EXERCISE 1.2.7. Draw the zero set of $x(x^2-\mu)(2x^2-\mu)$. What is the matter with the eigenvalue?

So far we concentrated on local results which can be proved in a constructive manner. But one can use topological non-constructive tools such as *degree theory,* to prove that the local branches cannot cease to exist and therefore should have a global continuation in some sense. More precisely they either have to set down in another bifurcation point or they should tend to infinity in the sense that either the x or the μ or both components are unbounded.

EXERCISE 1.2.8. Draw the zero sets of

(a) $x(x^2 + \mu^2 - 1)$,

(b) $x(x^2 + \mu^2 - 1)(x-\mu)$ *(secundary bifurcation),*

(c) $x(x + 1 - \frac{1}{\mu+1})$,

and interpret the results in the light of the remarks above.

Let $\xi \in \mathbb{R}^N$ be such that $\gamma(t) = x(t,\xi)$ is a periodic orbit with minimal period $p>0$. Linearization of the dynamical system $x(t,x_0)$ amounts to taking the derivative with respect to the initial condition x_0, and so we are led to consider the variational problem

$$\begin{cases} \dfrac{dH}{dt} = Df(\gamma(t))H \\ H(0) = I \end{cases}$$

where the $N \times N$ matrix $H(t)$ is $\frac{\partial x}{\partial x_0}(t,\gamma(0))$. Of particular interest are the eigenvalues of $H(p)$, which are called *Floquet multipliers* (or also *characteristic multipliers).* The fact that one of these eigenvalues has to be 1 reflects the translation invariance of the periodic orbit and can be proved by differentiation of the original nonlinear equation with respect to t. Indeed

$$\gamma''(t) = Df(\gamma(t))\gamma'(t)$$

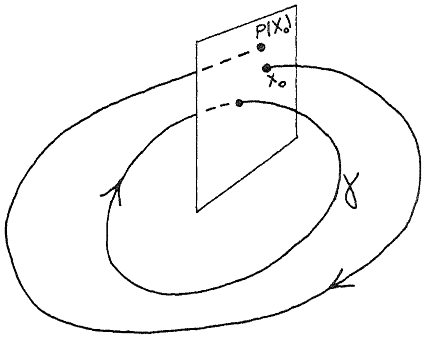
implies that $\gamma'(t) = H(t)\gamma'(0)$ and consequently $f(\gamma(0)) = H(p)f(\gamma(0))$. The eigenvalue 1 of $H(p)$ is called the trivial multiplier.

THEOREM 1.2.9. *If all nontrivial Floquet multipliers lie strictly inside the unit circle in the complex plane, the periodic orbit γ is exponentially stable with asymptotic phase. If at least one of them lies strictly outside the unit circle, the periodic orbit γ is unstable.*

Remarks 1.2.10. (i) One can show that H has a representation

$$H(t) = K(t)e^{Dt}$$

where K is p-periodic with $K(0) = I$, and D is constant. The eigenvalues of D are called *Floquet* (or *characteristic*) *exponents,* but one should realize that only the real parts of these are uniquely defined.
(ii) Let Π denote an $(N-1)$-dimensional subspace transverse to $f(\gamma(0))$, i.e. $\mathbb{R}^N = \Pi \oplus \text{span}\,\{f(\gamma(0))\}$. One can uniquely define a smooth function $\tau(x_0)$, for x_0 in a neighbourhood of $\gamma(0)$, such that $x(\tau(x_0),x_0)\in\Pi$ and $\tau(\gamma(0)) = p$.



The mapping $P:\Pi\to\Pi$ defined by

$$P(x_0) = x(\tau(x_0),\, x_0)$$

is called the *Poincaré map* (or *return map).* Clearly $\gamma(0)$ is a fixed point of P. One can show that the linearization of P at $\gamma(0)$ has precisely the $(N-1)$ nontrivial Floquet multipliers as its eigenvalues.

Periodic solutions experience certain bifurcations when their stability character changes, but the situation is essentially more complicated than in the case of steady states. From a practical point of view there is the added difficulty that one usually cannot calculate the periodic solutions and their Floquet multipliers explicitly. Nevertheless it is useful to know that

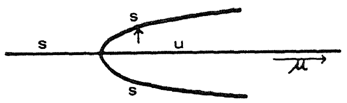
i) if a multiplier leaves or enters the unit circle through -1 the old periodic solution gets company of a new one with a period which is about twice as large. Such *period doublings* may occur repeatedly and even accumulate.

ii) if a pair of complex multipliers leaves or enters the unit circle then, as a rule, an *invariant torus* comes into existence. Such a torus can be "filled" with quasi-periodic orbits.

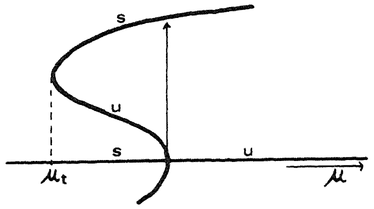
Admittedly this is a very incomplete and vague description of a huge collection of fascinating mathematical results which have recently been established and we refer the interested reader to Guckenheimer & Holmes (1983) for a detailed account.

1.3. An impressionistic sketch of some global aspects.

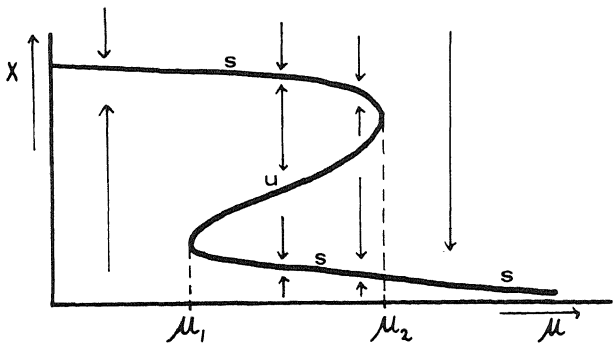
Bifurcations may produce smooth transitions which are hardly recognizable as in



or they can lead to large *(catastrophic)* transition as in



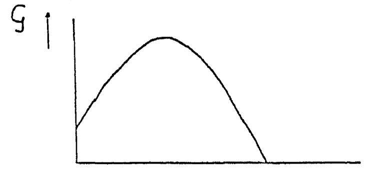
(Note that in *turning-points* such as μ_t the stability character of the steady states on one and the same branch may change; see Example 1.2.4 (v)). A situation which occurs rather frequently is an S - or Z - shaped branch:



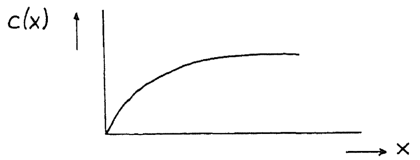
Example 1.3.1. In the stimulating paper May (1977), R.M. May discusses the following simple pedagogical example. Consider the biomass of vegetation (say grass) as a dynamical variable x. In the absence of herbivores the dynamics of x is assumed to be given by

$$\frac{dx}{dt} = G(x)$$

with G a function that can be typified graphically as follows:



Here $G(0)$ is taken to be positive in order to account for the supply of wind-borne seeds from other areas; the decline is due to mutual shading and competition for nutrients and K is the carying capacity. Next, introduce a population of herbivores which is kept at a constant density μ. The grazing is described by the per capita consumption rate $c(x)$



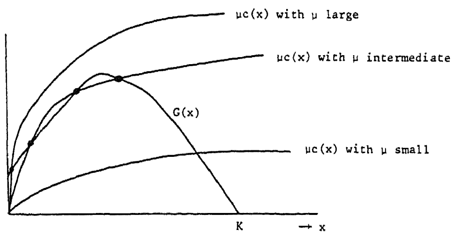
which is proportional to x at low x but which saturates at high x. So now

$$\frac{dx}{dt} = G(x) - \mu c(x)$$

and steady states have to satisfy the scalar equation

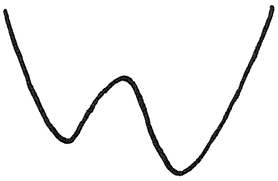
$$G(x) = \mu c(x).$$

The roots are easily found from graphical considerations:

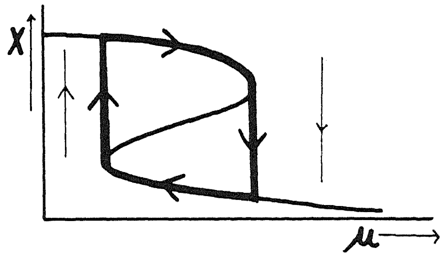


For large and small μ there exists only one steady state but for values in between we find three steady states (note that here we consider the case in which the c-curve saturates long before the G-curve reaches its top or, in other words, the nonlinearities have to be adjusted to each other in a specific manner in order to produce the Z).

For parameter values in between the turning points μ_1 and μ_2 the final state in which the system will settle down strongly depends on the *initial state* (say, at the beginning of spring). The domains of attraction of the stable upper and lower states are separated by the unstable state. In terms of a marble under the influence of gravity the landscape looks like:



Alternatively one can say that sufficiently large perturbations of the state x may produce a transition from the lower to the upper state or vice versa. What exactly is "sufficient" depends on the exact value of μ: a typical feature of the Z-curve is the shrinking and swelling of the domains of attraction with variations in the parameter μ. Indeed, another way to produce such transitions is to make μ undergo variations which include a trip outside the (μ_1,μ_2) interval. Note that such μ-variations are accompanied by a *hysteresis effect:* the steady state in which we will find the system for a given $\mu \in (\mu_1,\mu_2)$ depends on the *road* along which μ came to its present value. If μ is itself a slowly varying dynamical variable (i.e., a time scale argument has been used to uncouple the μ-dynamics and the x-dynamics) this may even lead to oscillations in which slow, gradual changes are alternated by fast, catastrophic transitions.



See the aforecited paper by MAY or see LUDWIG, JONES & HOLLING (1978) for an application of these ideas to the spruce- budworm (here x is the budworm density and μ^{-1} a measure for the leaf area).

The moral of the tale is that both chance and necessity play a leading part: the events are (assumed to be) predictable on the basis of deterministic relations between cause and effect, but nevertheless initial states and parameter values which, possibly, are subject to accidental perturbations (caused by forces or circumstances which we deliberately have chosen *not* to include in the model) have a decisive influence on what will actually happen.

Moreover the Z-curve illustrates the "grasp all, lose all" catastrophy: if in Example 1.3.1 one makes the cattle density μ larger and larger the domain of attraction of the economic desirable upper state becomes smaller and smaller until, for $\mu = \mu_2$, it is negligable and the least perturbation causes a catastrophic collaps of the grass-density. If, subsequently, the cattle density is lowered again the system will *not* return to the upper state but instead remain in th lower one! In other words: continuous changes in parameters may lead to discontinuous changes in steady states an the restoration of the old situation is *not* an immediate consequence of undoing the parameter change. The analogy with the consequences of over harvesting in a fishery suggests itself.

As a finale to this section we present some variations on a theme composed by C.S. Holling (HOLLING, 1973). The message is that putting a strong link between our mathematical stability concepts and our intuitive 'biological' ideas about ecosystem stability may be hazardous.

The mathematical stability of an equilibrium refers to the ability to recover from a temporary perturbation of the state variable(s). One can describe it in terms of the rate of return and the period of the oscillations that may accompany the return (or, in other words, the dominant eigenvalue). Therefore we are inclined to characterize a natural, real world, system that shows an almost constant appearance in the course of time as very stable. However, this constant appearance may be highly deceptive, by which we mean that it may be more the result of the almost complete absence of significant perturbations of both the state and the parameters.

As an example one can think of the great American lakes. Climatologic fluctuations are damped out by the water and the mobility of water animals guarantees a more or less homogeneous spatial distribution. For long times the composition of the inhabitant populations has been almost constant, a seemingly very stable situation. But human influence has led to radical changes. The supply of nutrients by waste water deposit and a strong harvesting pressure on certain fish populations has had a profound influence: some species became extinct and others were superseded by competitors which were better adapted to the new circumstances. There was no *resilience* and the disturbances created a new situation which was essentially different from the old.

On the other side of the spectrum one finds ecosystems which every now and then make large excursions through their state space (triggered by forest-fires, hurricanes or major pest outbreaks of short duration, like those of the spruce budworm) but which, maybe exactly while doing so, are very *resilient* in the sense that the overall pattern of relations between state-variables is essentially unaffected by many kinds of disturbances. Thus Holling argues that a strong time-inhomogeneity may enhance resilience and that the same may be true for spatial inhomogeneity (see the paper by SABELIS & LAANE in part B). Holling questions the appropriateness of the usual stability concepts for many types of ecosystems and promotes the resilience concept which heavily appeals to the intuition but seems hard to formalize in a strict mathematical sense (because it is neither clear what kind of disturbances one should consider (allow) nor what properties orbits should have). In the resilience concept at least two mathematical concepts, viz. dynamical stability and structural stability, are intertwined. The practical intuition on which it is based seems to be that in many (families of) systems a fast return rate to the equilibrium (or periodic solution) is linked to a large basin of attraction of that equilibrium (or periodic solution) and also to being far away (in parameter space) from a bifurcation point. Formalizing this notion presents one of the challenges of theoretical ecology!

2. An example of the construction of a dynamical system: an epidemic model with temporary immunity

2.1. The model.

In subsection IV.4.1 we considered the time course of an infectious disease within a closed population under the assumption that the disease conveys ever lasting immunity. We now drop this latter assumption and assume instead that an infected individual returns a fixed time T after its contagion to the class of susceptibles. After a scaling of time we may put $T = 1$. Hence $S(t)$, the number of susceptibles at time t, satisfies

$$S(t) = N - \int_0^1 i(t,a)da \tag{2.1}$$

where N denotes the total number of individuals and $\int_{a_1}^{a_2} i(t,\alpha)d\alpha, 0<a_1<a_2<1$, the number of individuals which were infected between $t-a_2$ and $t-a_1$. Let $A(a)$ denote the infectivity as a function of the time a elapsed since contagion. Of course we assume that $A(a) = 0$ for $a \geqslant 1$. The law of mass action assumption that the number of new cases equals the product of S and $\int_0^1 A(a)i(t,a)da$ leads to the problem

$$\begin{cases} \dfrac{\partial i}{\partial t} + \dfrac{\partial i}{\partial a} = 0 \\ i(t,0) = (N - \int_0^1 i(t,\alpha)d\alpha) \int_0^1 A(\alpha) i(t,\alpha) d\alpha \\ i(0,a) = \phi(a), \quad 0 \leqslant a \leqslant 1, \end{cases} \tag{2.2}$$

where the initial condition ϕ has to satisfy the *constraint*

$$\int_0^1 \phi(\alpha)d\alpha \leqslant N \tag{2.3}$$

(i.e., the number of infected individuals present at time $t=0$ is necessarily smaller than the total number present).

EXERCISE 2.1. Interpret the relation $\frac{dS}{dt}(t) = i(t,1) - i(t,0)$.

Let B_1 denote the indicator function of the interval [0, 1] i.e.

$$B_1(a) = \begin{cases} 1 & \text{for } 0 \leqslant a \leqslant 1 \\ 0 & \text{elsewhere} \end{cases} \tag{2.4}$$

The scaling

$$\begin{cases} i(t,a) = N\, n(t,a) \\ B_2(a) = \frac{N}{\gamma} A(a) \quad \text{with } \gamma = N \int_0^1 A(\alpha) d\alpha \end{cases} \tag{2.5}$$

yields

$$\begin{cases} \frac{\partial n}{\partial t} + \frac{\partial n}{\partial a} = 0 \\ n(t,0) = \gamma \left(1 - \int_0^1 B_1(\alpha) n(t,\alpha) d\alpha\right) \int_0^1 B_2(\alpha) n(t,\alpha) d\alpha \\ n(0,a) = \psi(a) \end{cases} \tag{2.6}$$

where the scaled initial condition ψ satisfies the constraint

$$\int_0^1 \psi(\alpha) d\alpha \leqslant 1 \, . \tag{2.7}$$

EXERCISE 2.2. Give a biological interpretation of γ (see subsection IV.4.1).

Defining $y(t) = n(t,0)$ we find by integration along characteristics that y has to satisfy

$$y(t) = \gamma \left[1 - \int_0^t B_1(a) y(t-a) da - f_1(t) \right] \left[\int_0^t B_2(a) y(t-a) da + f_2(t) \right] \tag{2.8}$$

where

$$f_1(t) = \int_t^1 B_1(a) \psi(a-t) da = \begin{cases} \int_0^{1-t} \psi(a) da & , \text{ for } t \leqslant 1 \\ 0 & , \text{ elsewhere} \end{cases} \tag{2.9}$$

$$f_2(t) = \int_t^1 B_2(a)\, \psi(a-t) da = \begin{cases} \int_0^{1-t} B_2(a+t) \psi(a) da & , \text{ for } t \leqslant 1 \\ 0 & , \text{ elsewhere} \end{cases} \tag{2.10}$$

Readers who are not interested in the proof of the existence and the uniqueness of a solution may pass from here direct to Section 2.3.

The fact that naturally two forcing functions arise suggests to reformulate (2.8) as a two dimensional system. So define

$$x_i(t) = \int_0^t B_i(a) y(t-a) da + f_i(t) \, , \quad i = 1,2 \, , \tag{2.11}$$

then

$$x(t) = \int_0^t K(a)\, g(x(t-a))da + f(t)\,, \tag{2.12}$$

where

$$K(a) = \begin{bmatrix} B_1(a) & 0 \\ 0 & B_2(a) \end{bmatrix} \tag{2.13}$$

and

$$g(x) = \begin{bmatrix} \gamma(1-x_1)\, x_2 \\ \gamma(1-x_1)\, x_2 \end{bmatrix} \tag{2.14}$$

Note that

$$y(t) = \gamma(1-x_1(t))\, x_2(t)\,. \tag{2.15}$$

2.2. Existence and uniqueness.

The main tool for the proof of the existence and uniqueness of solutions is the well known *contraction mapping principle.*

DEFINITION 2.3. Let X be a Banach space, U a closed subset and F a mapping from U to X. F is called a (strict) *contraction* on U if $\lambda \in [0,\, 1)$ exists such that

$$\|F(u)-F(v)\| \leqslant \lambda\, \|u-v\| \quad \text{for all } u,v \in U\,.$$

An element $u \in U$ is called a *fixed point* of F if $F(u) = u$.

THEOREM 2.4. (Banach-Cacciopoli). *If $F: U \to U$ is a contraction then F has a unique fixed point $\bar{u} \in U$; for any $u \in U$ the sequence $F^n(u)$ converges to $\bar{u}$ and, moreover,*

$$\|F^n(u)-\bar{u}\| \leqslant \lambda^n \frac{\|u-\bar{u}\|}{1-\lambda}\,.$$

DEFINITION 2.5. Let Λ be a set. A mapping $F: U \times \Lambda \to U$ is called a *uniform contraction* on U if $\lambda \in [0,1)$ exists such that

$$\|F(u,\rho)-F(v,\rho)\| \leqslant \lambda \|u-v\| \text{ for all } u,v \in U \text{and all } \rho \in \Lambda.$$

THEOREM 2.6. R (The uniform contraction principle). Let U and V be open subsets of Banach spaces X and Y and let $\bar{U}$ denote the closure of U. Suppose $F: \bar{U} \times V \to \bar{U}$ is a uniform contraction. Let $h = h(y)$ be the unique fixed point of $F(\cdot,y)$ in $\bar{U}$. If $F \in C^k(\bar{U} \times V, X), 0 \leqslant k < \infty$ then $h \in C^k(V,X)$.

Assume that B_2 is a bounded measurable function. For any $\theta > 0$ let $C_\theta = C[0,\theta] = C([0,\theta];\mathbb{R}^2)$ and define $F: C_\theta \to C_\theta$ by

$$F(x)(t) = \int_0^t K(a) g(x(t-a))da + f(t) \tag{2.16}$$

where f is a given element of C_θ.

EXERCISE 2.7. Verify that F indeed maps C_θ into itself.

We want to find a fixed point of F. Let

$$U = \{x|\ \|x-f\|_{C_\theta} \leqslant 1\}$$

Since $\|x\|_{C_\theta} \leqslant \|f\|_{C_\theta} + 1$ for all $x \in U$ we are led to introduce the Lipschitz constant L of g on the set $\{\xi|\ |\xi| \leqslant \|f\|_{C_\theta} + 1\}$, that is, $|g(\eta)-g(\xi)|_{\mathbb{R}_2} \leqslant L|\eta-\xi|_{\mathbb{R}_2}$, $\forall \eta,\xi$ in the ball of radius $\|f\|_{C_\theta} + 1$ in $\mathbb{R}_2$.

Putting

$$\|K\| = \max\{\sup B_1, \sup B_2\}$$

we clearly have for all $x,z \in U$

$$\|F(x)-F(z)\|_{C_\theta} \leqslant L\theta\|K\| \, \|x-z\|_{C_\theta} ,$$

and for θ sufficiently small $L\theta\|K\|<1$. Moreover,

$$\|F(x)-f\|_{C_\theta} \leqslant L\theta\|K\| \, \|x\|_{C_\theta} \leqslant L\theta\|K\| \, (1 + \|f\|_{C_\theta}) \leqslant 1$$

for θ sufficiently small. We conclude that for θ sufficiently small F is a contraction of U into U and consequently Theorem 2.4 implies that F has a unique fixed point in U.

EXERCISE 2.8. Use Theorem 2.6 to show that the fixed point depends continuously on the function f.

In the proof above the upper bound for θ depends on $\|f\|_{C_\theta}$. So if we repeat the argument to continue the solution to a larger time interval we have to investigate whether or not the new functions f will "blow-up" such that the intervals (the new upper bounds for θ) become smaller and smaller and do not cover the positive time axis as in the

EXAMPLE 2.9. The ordinary differential equation $\dot{y} = y^2$ has a solution $y(t) = -(t-c)^{-1}$ which explodes at the point $t = c$; the solution cannot be continued beyond the point $t = c$.

Now notice that the natural constraints $\int_0^1 \psi(a)da \leqslant 1$ and $\psi \geqslant 0$ imply that

$$0 \leqslant f_1(t) = \int_t^1 B_1(a)\psi(a-t)da \leqslant 1 \text{ and } 0 \leqslant f_2(t) = \int_t^1 B_2(a)\psi(a-t)da \leqslant \sup B_2 .$$

So we have *a priori bounds* for $\|f\|_{C_\theta}$ provided we demonstrate that the constructed solution, in terms of $n(t,\cdot)$, belongs to

$$C = \{\psi \in L_1(0,1) \mid \psi \geqslant 0 \text{ a.e. and } \int_0^1 \psi(a)da \leqslant 1\} . \tag{2.17}$$

Or, in other words, we have to show that C is (positively) *invariant* (as it should be on account of the biological interpretation). Since $\int_0^1 n(t,a)da = x_1(t)$ and $n(t,0) = y(t) = \gamma(1-x_1(t))x_2(t)$ we have to prove that $0 \leqslant x_1(t) \leqslant 1$ and $x_2(t) \geqslant 0$.

LEMMA 2.10. *If $f_1(0) \leqslant 1$ and $f_i(t) \geqslant 0$ for $i = 1,2$, then the solution of (2.12) satisfies $0 \leqslant x_1(t) \leqslant 1$ and $x_2(t) \geqslant 0$ as long as it is defined.*

PROOF. Assume first that $f_1(0)<1$. Let $\bar{t} = \sup\{t \mid x_1(t)<1\}$ and suppose that $\bar{t}<\infty$. By continuity $x_1(\bar{t}) = 1$. On $[0,\bar{t}]$, $x_2(t)$ is nonnegative and bounded from above (indeed, if we consider x_1 as known then x_2 can be obtained by monotone iteration: $x_2(t) = f_2(t) + \int_0^t B_2(a)\,(1-x_1(t-a))f_2(t-a)da + \cdots$, which shows nonnegativity; boundedness follows likewise from the inequality $x_2(t) \leqslant \gamma \int_0^t B_2(a)x_2(t-a)da + f_2(t))$. Differentiation of the equation for x_1 yields

$$\frac{dx_1}{dt} = \gamma(1-x_1)x_2 - h$$

where

$$h(t) = \begin{cases} \psi(1-t) & , t \leqslant 1 , \\ \gamma(1-x_1(t-1))x_2(t-1) & , t>1 . \end{cases}$$

Putting $z = 1-x_1$ we obtain

$$\frac{dz}{dt} = -\gamma z x_2 + h \geqslant -\gamma z x_2 \quad \text{for } t \in [0,\bar{t}]$$

$$\Rightarrow \ln\frac{z(t)}{z(0)} \geqslant -\gamma \int_0^t x_2(\tau)d\tau \Rightarrow z(t) \geqslant z(0)e^{-\gamma\int_0^t x_2(\tau)d\tau} > 0$$

for $t \in [0,\bar{t}]$ which is in contradiction with $z(\bar{t}) = 0$. So we conclude that $f_1(0) < 1$ implies that $x_1(t) < 1$. Invoking the continuous dependence on f we deduce that $f_1(0) \leqslant 1$ implies that $x_1(t) \leqslant 1$. □

SUMMARY: Take any $\psi \in C$. Choose θ such that $L\theta \|K\|(2 + \sup B_2) \leqslant 1$ where L is the Lipschitz constant of g in the ball of radius $2 + \sup B_2$. Then (2.12) has a unique solution in C_θ. Define for $0 \leqslant t \leqslant \theta$

$$n(t,a) = \begin{cases} \psi(a-t) & ,a \geqslant t \\ \gamma(1-x_1(t-a))x_2(t-a) & ,\ t > a. \end{cases} \tag{2.18}$$

Lemma 2.10 and equation (2.18) imply that $n(\theta,\cdot) \in C$. So with $n(\theta,\cdot)$ as a new initial condition we can construct a solution for $\theta \leqslant t \leqslant 2\theta$. Proceeding in this manner we obtain a unique solution defined for all $t \geqslant 0$. Thus we have associated a dynamical system with the problem (2.6) on C, the biologically interpretable subset of $L_1[0,1]$ (note that the continuity of the map $\psi \mapsto f$ together with the continuous dependence on f guarantees the continuous dependence on ψ).

REMARK. Without the restriction $\psi \in C$ solutions might indeed blow up in finite time.

EXERCISE 2.11. Verify that the infinitesimal generator is given by

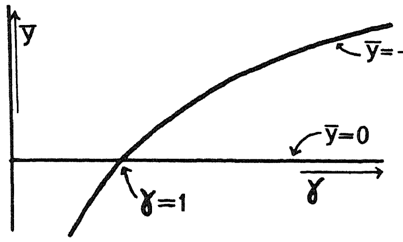
$$A\psi = -\psi'$$

$$\mathcal{D}(A) = \{\psi \in C \mid \psi \text{ is absolutely continuous and } \psi(0) = \gamma(1 - \int_0^1 \psi(a)da) \int_0^1 B_2(a)\psi(a)da\}\ .$$

Herewith the first step of the program is completed. The next step is concerned with finding the steady states and determining their stability.

2.3. *The stability of the steady states.*

EXERCISE 2.12. Show that steady states are given by $\bar{n}(a) = \bar{y}$ where either $\bar{y} = 0$ or $\bar{y} = \frac{\gamma-1}{\gamma}$. The second one is biologically relevant (i.e. nonnegative) if and only if $\gamma > 1$!



EXERCISE 2.13. Put $n(t,a) = \bar{y} + \rho(t,a)$ in (4.6) and neglect higher order terms in ρ to arrive at

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial a} = 0$$

$$\rho(t,0) = \begin{cases} \gamma \int_0^1 B_2(a)\rho(t,a)da \ \text{ for } \bar{y} = 0 \\[2ex] \int_0^1 \{B_2(a) + (1-\gamma)B_1(a)\}\rho(t,a)da \ \text{ for } \bar{y} = \frac{\gamma-1}{\gamma}\,. \end{cases} \tag{2.19}$$

EXERCISE 2.14. Derive the characteristic equation for $\bar{y} = 0$ and show that all roots lie in the left half plane for $\gamma < 1$ but that at least one root lies in the right half plane if $\gamma > 1$ (consult subsection IV.4.1). Convince yourselve that this result is trivial from a biological point of view by recalling the interpretation of γ and the threshold phenomenon discussed in subsection IV.4.1.

Since the principle of linearized stability is rigorously proved for systems of Volterra convolution integral equations like (2.12) (see, for instance, DIEKMANN & VAN GILS, 1984), Exercise 2.14 implies that $\bar{y} = 0$ is locally stable for $\gamma < 1$ and unstable for $\gamma > 1$. But in fact we can prove

THEOREM 2.15. *$\bar{y}$ is globally exponentially stable for $\gamma < 1$.*

PROOF. Equation (2.8) and the fact that both y and f_1 are nonnegative imply that $y(t) \leqslant \gamma \int_0^t B_2(a) y(t-a) da + \gamma f_2(t)$. Let z be the solution of the corresponding equation

$$z(t) = \gamma \int_0^t B_2(a) z(t-a) da + \gamma f_2(t) .$$

We want to show that $y(t) \leqslant z(t)$ for all t. Let us first assume that $f_1(0) > 0$. Then $y(0) < z(0)$. Let $\bar{t} = \sup\{t \mid z(t) > y(t)\}$ and assume that $\bar{t} < \infty$. Then necessarily $y(\bar{t}) = z(\bar{t})$. But on the other hand

$$z(\bar{t}) = \gamma \int_0^{\bar{t}} B_2(a) z(\bar{t}-a) da + \gamma f_2(\bar{t}) > \gamma \int_0^{\bar{t}} B_2(a) y(\bar{t}-a) da + \gamma f_2(\bar{t}) \geqslant y(\bar{t})$$

which is a contradiction. We conclude that $f_1(0) > 0$ implies that $y(t) < z(t)$, $\forall t \in [0, \infty)$. The continuous dependence on f then implies that always $y(t) \leqslant z(t)$. In subsection IV.2.2 it was shown that z converges exponentially to zero for $t \to \infty$. Since $n(t,a) = y(t-a)$ for $t > 1$ this proves the theorem. □

The principle of the exchange of stability tells us that $\bar{y} = \frac{\gamma-1}{\gamma}$ is stable for $\gamma > 1$ but $\gamma - 1$ small. In the present case this can also be easily verified by analysing the characteristic equation with the aid of the *implicit function theorem.*

THEOREM 2.16. *Let X, Y and Z be Banach spaces and $U \subset X$ and $V \subset Y$ open sets. Let F be a C^1-mapping from $U \times V$ into Z and let $(x_0, y_0) \in U \times Z$ be such that*

(i) $F(x_0, y_0) = 0$

(ii) $D_x F(x_0, y_0)$ *is an isomorphism (one-to-one and onto).*

Then there exist a neighbourhood $U_1 \times V_1$ of (x_0, y_0) and a function $f: V_1 \to U_1$ with $f(y_0) = x_0$ such that $F(x,y) = 0$ for $(x,y) \in U_1 \times V_1$ if and only if $x = f(y)$. If $F \in C^k$, $k \geqslant 1$, then $f \in C^k$.

EXERCISE 2.17. Show that the characteristic equation for $\bar{y} = \frac{\gamma-1}{\gamma}$ is

$$\bar{B}_2(\lambda) + (1-\gamma) \bar{B}_1(\lambda) = 1 \tag{2.20}$$

where $\bar{B}_i(\lambda) = \int_0^\infty e^{-\lambda\tau} B_i(\tau) d\tau$ is the Laplace transform of B_i.

EXERCISE 2.18. Verify that for $\gamma = 1$ (2.20) has the simple root $\lambda = 0$ while all other roots have strictly negative real part. Use the implicit function theorem to show that a unique root $\lambda(\gamma)$ exists with $\lambda(1) = 0$ and that $\lambda(\gamma) < 0$ for $\gamma > 1$.

Since roots can neither come out of the blue nor enter the right half plane at infinity (since $\bar{B}_i(\lambda) \to 0$ for $|\lambda| \to \infty$ with Re $\lambda \geqslant 0$) all roots have to lie in the left half plane when $0 < \gamma - 1 << 1$. We conclude that $\bar{y} = \frac{\gamma-1}{\gamma}$ is *locally stable* for those values of γ. G. Gripenberg (1981) has derived sufficient conditions for the *global* stability of $\bar{y} = \frac{\gamma-1}{\gamma}$ (in the sense that any initial condition in C different from $\psi \equiv 0$ yields a solution which for $t \to \infty$ converges to $\bar{y} = \frac{\gamma-1}{\gamma}$).

QUESTION: Does $\bar{y} = \frac{\gamma-1}{\gamma}$ remain stable when γ is further increased?

As remarked above $\bar{y} = \frac{\gamma-1}{\gamma}$ can only loose its stability if roots cross the imaginary axis. The substitution $\lambda = 0$

into (2.20) leads to $\gamma = 1$ so in fact only couples of complex conjugated roots can possibly cross the imaginary axis and equivalently we may ask the

QUESTION: Do Hopf bifurcations occur if γ increases from 1 to, say, ∞?

In order to investigate whether for certain values of γ roots of (2.20) lie exactly *on* the imaginary axis one can take $\lambda = i\eta$ with $\eta \in \mathbb{R}$. From a real point of view we have two equations in two unknowns η and γ. Since the equations are linear in γ it is possible to eliminate γ and to derive one equation in the unknown η. By a miracle this nonlinear equation can be analysed completely in this special case (see DIEKMANN & MONTIJN (1982)). The outcome is summarized in

THEOREM 2.19. *When γ increases from 1 to ∞ exactly as many pairs of complex conjugated roots of (2.20) cross the imaginary axis as there are $n \in \mathbb{N}$ for which*

$$b_n = 2 \int_0^1 B_2(a)\sin(2\pi n a)da > 0 .$$

These are simple and go from left to right with a positive speed at a "height" between $2n\pi$ and $(2n+1)\pi$.

The local Hopf bifurcation theorem for systems of Volterra convolution integral equations is proved in DIEKMANN & VAN GILS (1984).

Unfortunately Theorem 2.19 does not exclude the possibility that several pairs cross simultaneously and in resonance (i.e. one being an integer multiple of the other). Fiedler (preprint) has proved a *global* Hopf bifurcation theorem which applies to the present problem.

EXERCISE 2.20. Use the implicit function theorem to show that for $\gamma \to \infty$ the roots converge to the points $\pm 2n\pi i$ and that the sign of b_n determines whether they come from the right- or the left half plane.

REMARK. Only the first bifurcating periodic solution can possibly be stable (in other words: the later ones are necessarily unstable). Numerically the first usually corresponds to the root with the smallest η (i.e. the smallest frequency and the largest period; also see the next section) but there are exceptional cases in which a smaller period comes first.

EXERCISE 2.21. Show that $b_1 > 0$ if the support of B_2 is contained in $[0,\frac{1}{2}]$ and check the following interpretation: if the immunity period is long compared to the period of infectivity then enlargement of the population density leads to a destabilization of the stationary endemic state $\bar{y} = \frac{\gamma-1}{\gamma}$; in this situation one can expect to see oscillations.

CONCLUDING REMARKS: In nonlinear problems one can usually exploit bifurcation theory to obtain some information about the dynamic behaviour. But especially questions concerning *global* aspects are very hard to answer by analytical means. The most promising approach seems to try to combine the outcome of *numerical experiments* with the rather abstract general (topological) *theory of dynamical systems and bifurcations.* Again we refer to Guckenheimer & Holmes (1983) and Golubitsky and Schaeffer (1984) for an outline of the main results and ideas. For the model of this section numerical experiments have not been performed and the results presented above are more or less all that is known to the present authors.

3. Hopf bifurcation in scalar nonlinear renewal equations and nursery competition.

3.1. Introduction to the theory.

In the preceeding subsection we encountered a system of two Volterra convolution equations (or, in other words, renewal equations) and we presented results concerning Hopf bifurcations. In this section we shall show in some more detail how such results can be derived by concentrating on the somewhat simpler problems which take the form of just one equation

$$x(t) = \int_0^\infty g(a)f(x(t-a))da \ , \quad x(t) \in \mathbb{R} \ , \tag{3.1}$$

where the (nonnegative) kernel g is normalized to have integral one, and where both g and the nonlinearity f may depend on parameters. Before becoming more specific we present some generalities.

Steady states of (3.1) are found from the equation

$$\bar{x} = f(\bar{x}) . \tag{3.2}$$

The linearization around a steady state $\bar{x}$ is given by

$$y(t) = f'(\bar{x}) \int_0^\infty g(a)y(t-a)da \tag{3.3}$$

and the corresponding characteristic equation reads

$$1 = f'(\bar{x})\, \bar{g}(\lambda) \tag{3.4}$$

where, as before, $\bar{g}$ denotes the Laplace transform. In population problems g usually is nonnegative which allows us to draw the following conclusions (compare subsection IV.2.2; note that $\bar{g}(0) = 1$):

(i) if $f'(\bar{x}) > 1$ there is a dominant positive real root and $\bar{x}$ is unstable;

(ii) if $0 < f'(\bar{x}) < 1$ there is a dominant negative real root and $\bar{x}$ is asymptotically stable; since the dominant root is real, typical trajectories approaching $\bar{x}$ will do so monotonically (i.e. without oscillating) and one calls $\bar{x}$ over-damped stable.

(iii) if $f'(\bar{x}) = 0$ there are no roots at all (if $f'(\bar{x})$ approaches zero, when some parameter is varied, roots will tend to ∞ in the left half plane of $\mathbb{C}$).

(iv) if $-1 \leqslant f'(\bar{x}) < 0$ all roots are complex and have negative real part (indeed, $|\bar{g}(\lambda)| < 1$ if $\mathrm{Re}\lambda \geqslant 0$, $\lambda \neq 0$, and therefore no root can lie in the right half plane); since now typical trajectories approaching $\bar{x}$ will oscillate one calls $\bar{x}$ under-damped stable.

(v) if $f'(\bar{x}) < -1$ all roots are still complex but the possibility that they lie in the right half plane exists; so $\bar{x}$ is either under-damped stable or unstable.

Next consider the situation that $\bar{x}$ depends on some parameter(s). The stability properties of $\bar{x}$ can change in two ways when the parameters are varied. The first is connected with the bifurcation of steady states and occurs when $\lambda = 0$ is a root (which will be the case if and only if $f'(\bar{x}) = 1$). The second is connected with the bifurcation of periodic solutions and occurs if $\lambda = \pm i\omega$ is a root for some $\omega \in \mathbb{R}$. This will happen if and only if

$$\mathrm{Im}\, \bar{g}(i\omega) = 0 \tag{3.5}$$

and

$$f'(\bar{x}) = (\mathrm{Re}\, \bar{g}(i\omega))^{-1} . \tag{3.6}$$

In the following our approach will be to solve first (3.5) for ω (which will usually give countably many solutions ω_k) and subsequently analyse (3.6), with $\omega = \omega_k$, to find the parameter values for which roots lie exactly on the imaginary axis. Our strategy will be to concentrate on *two* parameters simultaneously, for the very simple reason that the curves in the plane which one gets as solutions of (3.6) suit the human physiological and psychological possibilities to take in information so very well.

As we have seen in subsection 1.2, the principle of the exchange of stability establishes a clear relation between the stability of the bifurcating periodic solutions and the direction of bifurcation. In DIEKMANN & VAN GILS (1984) an "explicit" formula is derived for the direction of bifurcation. The difference between "explicit" and explicit is that one still has to invert two matrices. In the present scalar case that is very easy and a straightforward application of Theorem 11.5 in DIEKMANN & VAN GILS (1984) yields:

THEOREM 3.1. *Consider a path in parameter space along which a simple root of (3.4) crosses the imaginary axis at ω from left to right with positive speed and assume that all other roots of (3.4) lie strictly in the left half plane. Let c_1 be defined by*

$$c_1 = -\frac{1}{2\,(f'(\bar{x}))^2 \bar{g}'(i\omega)} \left\{ f'''(\bar{x}) + 2\,\frac{(f''(\bar{x}))^2}{1-f'(\bar{x})} + \frac{(f''(\bar{x}))^2\, \bar{g}(2i\omega)}{1-f'(\bar{x})\bar{g}(2i\omega)} \right\} . \tag{3.7}$$

Then $\mathrm{Re}c_1 < 0$ *implies that the bifurcating periodic solution exists supercritically and is stable, whereas* $\mathrm{Re}c_1 > 0$ *implies that the bifurcating periodic solution exists subcritically and is unstable.*

3.2. A first application.

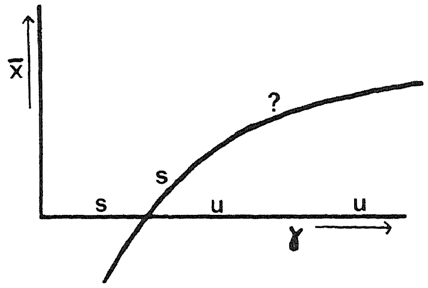
Our first application is a caricature of the competition equation derived in subsection III.6.2 (exercise 6.2.7 with $F(m) = -\log(1-m)$) in that we take explicit forms for both the kernel and the nonlinearity, basing ourselves on no other justification than the simplicity which results. So consider the equation

$$x(t) = \frac{\gamma}{2\epsilon} \int_{1-\epsilon}^{1+\epsilon} x(t-a) e^{-x(t-a)} da \,. \tag{3.8}$$

Here γ can be thought of as the expected number of offspring produced by a newborn individual during its entire life span in the absence of density dependent effects. The reproductivity is concentrated in a "window" of width 2ϵ centered at 1 (so time is scaled such that the midpoint of the reproductivity period, which can be thought of as a *generation time,* is reached exactly one time unit after birth). Finally, density dependence is incorporated in the exponential "correction" factor.

REMARK 3.2. Similar equations are used in human demography to describe the so-called Easterlin effect which says that women which are born during the bulge of a birth wave tend to get less children than those who are born in the lower valleys. See Swick (1981b).

Equation (3.8) has two steady states: $\bar{x} = 0$ and $\bar{x} = \ln\gamma$.



EXERCISE 3.3. Verify the stability assertions in the above diagram.

EXERCISE 3.4. Define $g(a) = \frac{1}{2\epsilon}$ for $1-\epsilon \leqslant a \leqslant 1+\epsilon$ and $g(a) = 0$ elsewhere. Show that the characteristic equation corresponding to the nontrivial steady state $\bar{x} = \ln\gamma$ is

$$1 = (1-\ln\gamma)\,\bar{g}(\lambda)\,. \tag{3.9}$$

EXERCISE 3.5. Show that $\bar{g}(i\omega) = \frac{\sin\epsilon\omega}{\epsilon\omega}[\cos\omega - i\sin\omega]$ and conclude that the roots of $\mathrm{Im}\,\bar{g}(i\omega) = 0$ are precisely the points $\omega = k\pi$, $k \in \mathbf{Z}$.

EXERCISE 3.6. Rewrite the equation $1 = (1-\ln\gamma)\,\mathrm{Re}\,\bar{g}(ik\pi)$ as

$$\ln\gamma = 1 + (-1)^{k+1} \frac{\epsilon k\pi}{\sin\epsilon k\pi} \tag{3.10}$$

Take $k = 1$ and draw $\ln\gamma$ as a function of ϵ (verify that $|\frac{\xi}{\sin\xi}| \geqslant 1$).

EXERCISE 3.7. Show that $|\frac{2\xi}{\sin 2\xi}| > |\frac{\xi}{\sin\xi}|$ for $\xi \neq 0$ and convince yourself that more generally $|\frac{k\xi}{\sin k\xi}| > |\frac{\xi}{\sin\xi}|$ for $\xi \neq 0$ and $k = 2,3,4,\cdots$. Now take $k = 2,3,4,\cdots$. and draw $\ln\gamma$ given by (3.10) as a function of ϵ for $0 \leqslant \epsilon < 1$. The result should look like figure 3.1.

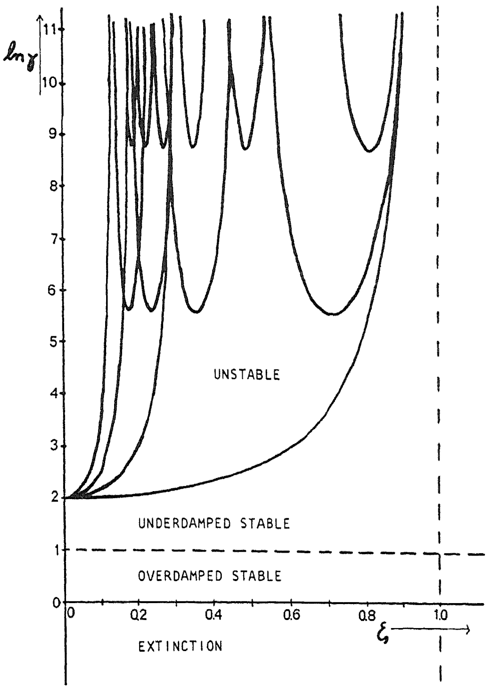


Figure 3.1. Parameter dependence of the qualitative behaviour of the solutions to (3.8).

Conclusion: There exists a curve in $(\ln\gamma,\epsilon)$-space given by $\ln\gamma = 1 + \frac{\epsilon\pi}{\sin\epsilon\pi}$ which is a *stability boundary* in the sense that it separates the region where $\bar{x}$ is stable from the region where $\bar{x}$ is unstable. One can pass this stability boundary either by shortening the reproductive period (decreasing ϵ) or by increasing the fertility (increasing γ)or by a combination of these two effects. At the stability boundary a Hopf bifurcation occurs and a periodic solution with a period of approximately two times the generation time arises.

We refer to Swick (1981b) for numerical experiments which indicate that the periodic solution may undergo a sequence of *period doublings* culminating in *chaotic behaviour* (in close analogy with the situation for the difference equation $x(t) = \gamma x(t-1)e^{-x(t-1)}$) when γ is further increased for some small value of ϵ.

In conclusion of our study of this simple example we shall try to determine the direction of the Hopf bifurcation.

Exercise 3.8. Show that

(i) $\bar{g}(2\pi i) = \frac{\sin\epsilon\pi}{\epsilon\pi}\cos\epsilon\pi$; note that $\bar{g}(2\pi i) \in \mathbb{R}$!

(ii) $\mathrm{Re}\,\bar{g}'(\pi i) = \frac{\sin\epsilon\pi}{\epsilon\pi} > 0$ for $0 \leqslant \epsilon < 1$.

(iii) $f'(\bar{x}) = 1-\ln\gamma$; $f''(\bar{x}) = \ln\gamma-2$; $f'''(\bar{x}) = 3-\ln\gamma$ and conclude that

$$\mathrm{sign}\,\mathrm{Re}\,c_1 = \mathrm{sign}\left\{\frac{\epsilon\pi}{\sin\epsilon\pi}-2-2\frac{\left(\frac{\epsilon\pi}{\sin\epsilon\pi}-1\right)^2}{\frac{\epsilon\pi}{\sin\epsilon\pi}+1}-\frac{\left(\frac{\epsilon\pi}{\sin\epsilon\pi}-1\right)^2\frac{\sin\epsilon\pi}{\epsilon\pi}\cos\epsilon\pi}{1+\cos\epsilon\pi}\right\}.$$

This expression is easily evaluated for $\epsilon = 0$ and $\epsilon = \frac{1}{2}$ and one finds $\mathrm{Re}\,c_1 < 0$, i.e. a supercritical stable bifurcation. But for $\epsilon \uparrow 1$, $\mathrm{Re}\,c_1 > 0$, so there must be at least one value of ϵ at which a change from supercritical to subcritical occurs. Numerically one finds there is exactly one at $\epsilon \approx 0{,}5536$. We have no biological interpretation for the switching from supercritical to subcritical.

3.3. *Nursery competition.*

Our second example is the model for competition in the nursery derived in section III.6.2. Consider

$$b(t) = \gamma \int_0^\infty g(a) b(t-a) e^{-(F(Q(t-a))-F(0))} da \tag{3.11}$$

where Q as a function of b is implicitly defined by

$$Q = b \frac{1-e^{-F(Q)}}{F(Q)} \tag{3.12}$$

and F is a strictly increasing function. In terms of the death rate ν and the fecundity function B the kernel g is

$$g(a) = \frac{1}{\gamma} B(a) e^{-\int_0^a \nu(\alpha) d\alpha} e^{-F(0)}, \tag{3.13}$$

where the parameter γ is chosen such that the integral of g is exactly one. Note that the seemingly superfluous insertion of the term $e^{-F(0)}$ allows us to give γ its usual interpretation of the expected number of offspring produced by a newborn individual during its entire lifespan in the absence of density dependent effects (the point is that the function F derived in Appendix III.6.A has $F(0)>0$ implying that even with all safe places available the rate of recruitment is different from the birth rate).

For $0<\ln\gamma<F(\infty)-F(0)$, (3.11) has a unique positive steady state

$$\bar{b} = \frac{\bar{Q}F(\bar{Q})}{1-e^{-F(\bar{Q})}} \tag{3.14}$$

where

$$\bar{Q} = F^{-1}(F(0) + \ln\gamma). \tag{3.15}$$

After linearization one obtains the characteristic equation

$$1 = [1-\bar{b}F'(\bar{Q})Q'(\bar{b})]\,\bar{g}(\lambda). \tag{3.16}$$

Exercise 3.9. Use (3.12) to deduce that

$$Q'(b) = \frac{1-e^{-F(Q)}}{F(Q) + F'(Q)[Q-be^{-F(Q)}]}$$

and use this expression to rewrite the coefficient of $\bar{g}(\lambda)$ in (3.16) as

$$\frac{F(\bar{Q}) + F'(\bar{Q})[\bar{Q}-\bar{b}]}{F(\bar{Q}) + F'(\bar{Q})[\bar{Q}-\bar{b}e^{-F(\bar{Q})}]} \tag{3.17}$$

We now consider two special functions F. The first is

$$F(Q) = \theta\frac{Q}{1+Q} \tag{3.18}$$

which is obtained from the cannibalism model discussed in the final remark of example III.6.2.3 by taking: (i) for Φ the Holling disk factor $\Phi(c) = \frac{1}{1+c}$; (ii) for c a constant times Q; (iii) a scaling of b and Q; (iv) $F(Q) = \theta Q\Phi(Q)$. The parameter θ is an indicator for the strength of the cannibalistic interaction (see Diekmann et al. (1986) for a more detailed interpretation).

Exercise 3.10. Show that now, for $0<\ln\gamma<\theta$,

$$\bar{Q} = \frac{\ln\gamma}{\theta-\ln\gamma} \tag{3.19}$$

$$\bar{b} = \frac{\gamma\ln^2\gamma}{(\gamma-1)(\theta-\ln\gamma)}$$

and derive for (3.17) the explicit expression

$$E(\theta,\gamma) = \frac{\theta(2\gamma-2-\gamma\ln\gamma)-\ln\gamma(\gamma-1-\gamma\ln\gamma)}{\theta(2\gamma-2-\ln\gamma)-\ln\gamma(\gamma-1-\ln\gamma)} \,. \tag{3.20}$$

EXERCISE 3.11. Solve the equation $E(\theta,\gamma) = -\Omega$ for θ in terms of γ and Ω to find

$$\theta = \ln\gamma\frac{\gamma-1-\gamma\ln\gamma + \Omega(\gamma-1-\ln\gamma)}{2\gamma-2-\gamma\ln\gamma + \Omega(2\gamma-2-\ln\gamma)} \tag{3.21}$$

We are interested in the function $\theta = \theta(\gamma)$ defined by (3.21) when

(i) $\Omega = 0$; this gives the boundary between the regions in parameter space where the steady state is overdamped stable and underdamped stable

(ii) $\Omega = -(\mathrm{Re}\bar{g}(i\omega))^{-1} > 1$; with ω such that $\mathrm{Im}\bar{g}(i\omega) = 0$; this gives curves in parameter space where a root of the characteristic equation (3.16) lies on the imaginary axis.

EXERCISE 3.10. Show that

$$\frac{d\theta}{d\Omega} = \left[\frac{(\gamma-1)\ln\gamma}{2\gamma-2-\gamma\ln\gamma + \Omega(2\gamma-2-\ln\gamma)}\right]^2 > 0$$

and conclude that the curve corresponding to the minimal positive value of Ω yields the boundary between the regions in parameter space where the steady state is stable and unstable.

EXERCISE 3.11. Show that the nominator of (3.21) has precisely one zero $\gamma^*(\Omega)$ in $(1,\infty)$ for $\Omega \geq 0$.

EXERCISE 3.12. Show that (3.21) satisfies the biological constraint $\theta > \ln\gamma$ if and only if $\gamma > \gamma^*(\Omega)$.

Figure 3.2A depicts the biologically relevant part of the graph of $\theta(\gamma)$ for $\Omega = 2.2618$. This value of Ω is obtained when $\nu(a) = \nu = 1$ and

$$B(a) = \begin{cases} 0 & \text{for } 0 \leq a \leq 1 \\ \gamma e & \text{for } a > 1. \end{cases}$$

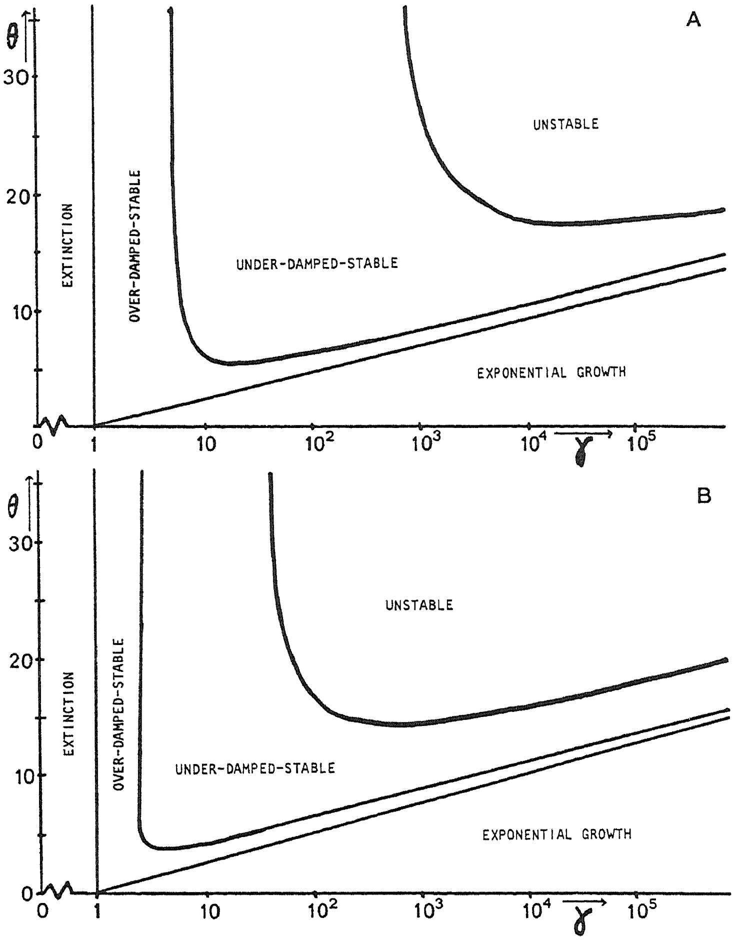


Figure 3.2. Stability diagram for the nursery competition model.
A: Cannibalism (formula (3.18)). B: Safe places (formula (3.22) with $\xi/\rho = 2.75$.)

The second example is

$$F(Q) = \frac{\theta}{2Q}\,\{Q-\xi + \sqrt{(Q-\xi)^2 + \rho Q}\} \tag{3.22}$$

which was introduced in Appendix III.6.A to describe competition for safe places.

EXERCISE 3.13. Show that $F(0) = \lim_{Q\downarrow 0} F(Q) = \frac{\rho\theta}{4\xi}$ and $F(\infty) = \theta$.

EXERCISE 3.14. Show that now

$$\bar{Q} = \frac{\xi \ln\gamma}{\ln\gamma + \frac{\rho\theta}{4\xi} - \frac{1}{\theta}(\ln\gamma + \frac{\rho\theta}{4\xi})^2} \tag{3.23}$$

$$\bar{b} = \frac{\xi\gamma\ln\gamma}{(1-\frac{1}{\theta}(ln\gamma + \frac{\rho\theta}{4\xi}))(\gamma - \exp(-\frac{\rho\theta}{4\xi}).} \tag{3.24}$$

Subsequently one can derive for (3.17) the explicit expression

$$E(\theta,\gamma) = \frac{c_1\theta^2 + c_2\theta + c_3}{c_4\theta^2 + c_5\theta + c_6} \tag{3.25}$$

where

$$\begin{cases} c_1 = -\gamma \\ c_2 = 2\gamma(F(0) + \ln\gamma) + (1-4\frac{\xi}{\rho})(\gamma - e^{-F(0)} - \gamma(F(0) + \ln\gamma)) \\ c_3 = -4\gamma\frac{\xi}{\rho}(F(0) + \ln\gamma)^2 \\ c_4 = -e^{-F(0)} \\ c_5 = 2e^{-F(0)}(F(0) + \ln\gamma) + (1-4\frac{\xi}{\rho})(\gamma - e^{-F(0)} - e^{-F(0)}(F(0) + \ln\gamma)) \\ c_6 = -4e^{-F(0)}\frac{\xi}{\rho}(F(0) + \ln\gamma)^2 , \end{cases} \tag{3.26}$$

and the equation $E(\theta,\gamma) = -\Omega$ can be reduced to the equation

$$\theta^2 + d_1\theta + d_2 = 0 \tag{3.27}$$

where

$$\begin{cases} d_1 = -\{(1+\Omega)(1-4\frac{\xi}{\rho})\left[\frac{\gamma e^{F(0)} - 1}{\gamma e^{F(0)} + \Omega}\right] - (1-4\frac{\xi}{\rho})(F(0) + \ln\gamma) + 2(F(0) + \ln\gamma)\} \\ d_2 = 4\frac{\xi}{\rho}(F(0) + \ln\gamma)^2. \end{cases} \tag{3.28}$$

Although $F(0) = \frac{\rho\theta}{4\xi}$ depends on θ, we may interpret (3.27) as a quadratic equation in θ after a change of variable $\zeta = \gamma e^{F(0)}$. Thus we find, for given Ω,ξ and ρ,θ as a double valued function of ζ (the condition that the roots should be real and positive leads to a lower bound for the allowed values of ζ). Subsequently one can recover θ as a function of γ by inversion of the $\gamma \mapsto \zeta$ transformation (i.e. $\gamma = \zeta\exp(-\frac{\rho\theta}{4\xi})$). The outcome of the numerical calculations (with the same value of Ω as in Figure 3.2A) is presented in Figure 3.2B.

4. Lyapunov functions and monotone methods: the G-M model in cell kinetics

In this section we discuss a model for the growth of a cell population. One of the basic assumptions is that the cell cycle consists of two phases. The model is a slight adaption of a model first suggested by Kirk, Orr and Forest (1970), which describes the production of red blood cells by the bone marrow stem cell population. Later this model has also been investigated by Mackey (1978, 1981).

In this section we shall use two basic techniques to derive global stability results for trivial and nontrivial equilibria namely Lyapunov functions (and the invariance principle) and monotonicity methods. Although some of the computations can be simplified, we think that the formulation below most clearly illustrates the underlying idea.

4.1. The model

We consider a population of cells reproducing by division. We assume that within the cell cycle at least two phases can be distinguished: The G-phase or resting phase during which cells "just sit and wait", and the M-phase or mitosis phase. We assume that a cell which has entered the M-phase finally passes into mitosis and its daughters enter the G-phase again which they leave after an exponentially distributed time. For more biological details concerning the life cycle of cells we refer to Eisen (1979).

We assume that all individuals in the G-phase are identical and we denote their number at time t by $P(t)$. However cells in the M-phase can be distinguished from one another according to some one-dimensional quantity x, which we shall call maturity, but which can be anything such as age, or the concentration of some chemical substance (like

DNA) within the cell. We let $n(t,x)$ be the maturity distribution, i.e. $\int_{x_1}^{x_2} n(t,x)dx$ is the number of M-cells with maturity between x_1 and x_2. A cell entering the M-phase has maturity $x = 0$. As in section I.4 and chapter II we conceive of fission as a stochastic process which can be described by a function $b(x) \geqslant 0$. We assume that the maximal maturity is $x = 1$, which is achieved mathematically by assuming that

$$\int_0^1 b(x)dx = \infty .$$

As in section I.4 and chapter II we assume that the maturity of an individual in the M-phase increases deterministically according to the ODE

$$\frac{dx}{dt} = V(x) ,$$

where $V(x)$ is called the growth rate. For simplicity we assume a constant death rate $\mu > 0$ for all individuals in the G-phase as well as in the M-phase. Finally we let γ be the transition probability, i.e. the chance per unit of time that cells in the G-phase enter the M-phase.

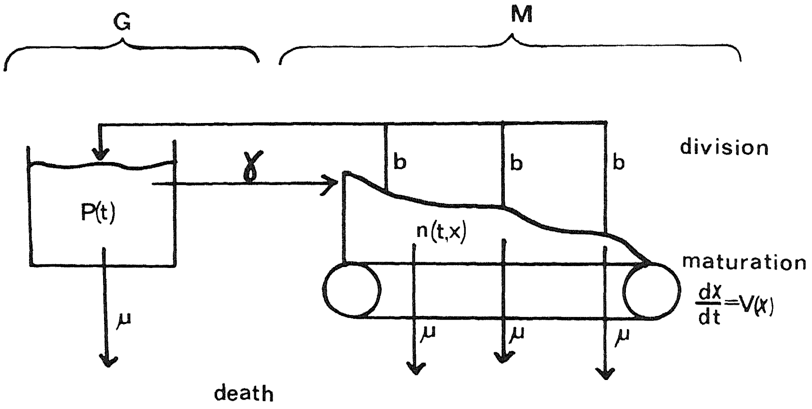


Figure 4.1.1. Schematic representation of the G-M model.

When μ, γ, V and b are all constant the dynamics of $P(t), n(t,x)$ is governed by the following linear system:

$$\frac{dP}{dt} = -\mu P - \gamma P + 2 \int_0^1 b(x)n(t,x)dx \tag{4.1}$$

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x} (V(x)n(t,x)) = -\mu n(t,x) - b(x)n(t,x) \tag{4.2}$$

$$V(0)n(t,0) = \gamma P. \tag{4.3}$$

EXERCISE 4.1. Convince yourself that (4.1) - (4.3) are correct and interprete the separate terms.

We supply (4.1) - (4.3) with the initial conditions

$$P(0) = P_0 \geqslant 0 \tag{4.4}$$

$$n(0,x) = n_0(x) \geqslant 0, \; 0 \leqslant x \leqslant 1 . \tag{4.5}$$

As in chapter II, we make the following

ASSUMPTIONS 4.2:

H_V: V is a strictly positive continuous function

H_b: b is non-negative and continuous on [0, 1) and $\lim_{x \uparrow 1} \int_a^x b(\xi)d\xi = \infty$.

Let $G(x), E(x)$ be defined as

$$G(x) = \int_0^x \frac{d\xi}{V(\xi)}, \tag{4.6}$$

$$E(x) = \exp(-\int_0^x \frac{\mu + b(\xi)}{V(\xi)} d\xi). \tag{4.7}$$

EXERCISE 4.3. Show that the distribution $n(t,x)$ can be expressed in terms of $P(t)$ in the following way

$$n(t,x) = \gamma P(t - G(x)) \frac{E(x)}{V(x)}.$$

EXERCISE 4.4. Use this result to show that the characteristic equation associated with (4.1) - (4.3) looks as follows:

$$\lambda + \mu + \gamma = 2\gamma \int_0^1 \frac{b(x)E(x)}{V(x)} e^{-\lambda G(x)} dx.$$

Show that this equation has exactly one real solution λ_d. Show that λ_d is strictly dominant, i.e. for all other roots of the equation the inequality $\mathrm{Re}\lambda < \lambda_d$ holds.

Of course, in practice the quantities μ, γ, V and b will depend on environmental factors such as food density, concentration of toxic chemicals etc. In this section we consider the very simple case where μ, V and b do not depend on such factors, and γ depends on the number of individuals in the G-phase $P(t)$.

ASSUMPTION 4.5. $\gamma = \gamma(P)$, γ is a Lipschitz-continuous function on $[0,\infty)$ which is decreasing, and moreover $\lim_{P \to \infty} \gamma(P) = 0$.

This means among other things that the chance that a cell in the G-phase enters the M-phase decreases if the size of the total G-population increases and that it approaches zero for $P \to \infty$.

It was first suggested by Kirk, Orr and Forest (1970), see also Mackey (1981), that the transition probability γ might depend on the concentration of some mitotic inhibitory substance. At the end of this section we shall indicate how this problem is related to assumption 4.5. See also Mackey (1978).

Assumption 4.5 makes (4.1) - (4.3) nonlinear, but keeps the system autonomous.

4.2. *Existence and uniqueness*

First we have to make some remarks on the problem of existence and uniqueness of solutions. As in chapter II we impose the following condition on the initial function n_0 (compare this to the result in Exercise 4.3):

ASSUMPTION 4.6. $n_0(\cdot)/E(\cdot)$ is a continuous function on $[0,1]$.

Now let X be the space of all pairs (ρ,ν) such that $\rho \in \mathbb{R}$ and $\frac{\nu(\cdot)}{E(\cdot)} \in C[0,1]$, and let X be supplied with the norm

$$\|(\rho,\nu)\| = |\rho| + \sup_{0 \leqslant x \leqslant 1} \frac{|\nu(x)|}{E(x)}.$$

EXERCISE 4.7. Verify that with this norm X defines a Banach space.

We define the cone X_+ in the following way:

$$\phi = (\rho,\nu) \in X_+ \text{ if and only if } \rho \geqslant 0 \text{ and } \frac{\nu(x)}{E(x)} \geqslant 0,\ x \in [0,1].$$

Now let $\phi_1 = (\rho_1,\nu_1)$, $\phi_2 = (\rho_2,\nu_2) \in X_+$ then $\phi_1 \leqslant \phi_2$ if $\phi_2 - \phi_1 \in X_+$, $\phi_1 < \phi_2$ if $\phi_2 - \phi_1 \in X_+$ and $\phi_2 \neq \phi_1$ and finally $\phi_1 << \phi_2$ if $\rho_1 < \rho_2$ and $\frac{\nu_1(x)}{E(x)} < \frac{\nu_2(x)}{E(x)}$ for all $x \in [0,1]$.

We define $X(t,x)$ as the maturity of an individual at time t given that its maturity at time zero was x. In other words $X(t,x)$ is the solution of

$$\frac{dX}{dt} = V(X(t,x)),\ X(0,x) = x\ ,$$

and $X(t,x)=G^{-1}(t+G(x))$, with $G(x)=\int\limits_0^x \frac{d\xi}{V(\xi)}$ (compare chapter II), provided that $0 \leqslant t < G(1)-G(x)$.

Let the initial pair $(P_0,n_0) \in X_+$ then $(P(t),n(t))$ is called a solution of (4.1) - (4.5) (with $\gamma = \gamma(P(t))$ substituted) if and only if

(i) $(P(t),n(t)) \in X_+,\ t \geqslant 0,$

(ii) $P(t)$ is differentiable for $t>0$ and

$$\frac{dP}{dt}(t) = -\mu P(t) - \gamma(P(t))P(t) + 2\int\limits_0^1 b(x)n(t,x)dx\ ,$$

(iii) $\lim\limits_{h\to 0}\frac{1}{h}\{V(X(h,x))n(t\ +\ h,\ X(h,x)) - V(x)n(t,x)\}$ exists for all $t>0$, $0<x<1$ and

$$\frac{1}{V(x)}\lim_{h\to 0}\frac{1}{h}\{V(X(h,x))n(t+h,X(t,x)) - V(x)n(t,x)\}\ =\ -\mu n(t,x) - b(x)n(t,x),\ \ t>0,\ 0<x<1\ ,$$

(iv) $V(0)n(t,0) = \gamma(P(t))P(t),\ t>0\ ,$

(v) $P(0) = P_0\ ,\ \ n(0,x) = n_0(x),\ 0 \leqslant x \leqslant 1\ .$

Condition (iii) means that n has to be differentiable along the characteristics.

As in Exercise 4.3 we can deduce

$$n(t,x) = \gamma(P(t-G(x)))P(t-G(x))\frac{E(x)}{V(x)}\ , \tag{4.8}$$

and if we substitute this in (4.1) we obtain

$$\frac{dP}{dt}(t) = -\mu P(t) - f(P(t)) + 2\int\limits_0^1 k(x)f(P(t-G(x)))dx\ ,$$

where

$$k(x) = \frac{b(x)}{V(x)}E(x),\ 0 \leqslant x < 1\ , \tag{4.9}$$

$$f(P) = \gamma(P)P\ ,\ \ P \geqslant 0\ . \tag{4.10}$$

From (4.8) we conclude that

$$n(t,x) = \frac{E(x)}{V(x)}\, f(P(t-G(x)),t>G(x)\ .$$

A similar calculation shows that

$$n(t,x) = \frac{E(x)}{V(x)}\frac{V(X(-t,x))}{E(X(-t,x))}\, n_0(X(-t,x)),\ t<G(x)\ .$$

Therefore at time $t = G(x)$, where $0<x<1$, $n(t,x)$ is discontinuous in x unless $f(P_0) = V(0)n_0(0)$. We define the subset C of X as

$$C = \{\phi = (\rho,\nu) \in X_+ | f(\rho) = V(0)\nu(0)\}\ .$$

Since the definition of a solution of (4.1) - (4.5) involves that $n(t,x)$ has to be continuous for $t>0$ we must start with initial pairs (P_0,n_0) belonging to C. We can then prove the following existence and uniqueness result.

THEOREM 4.8. *Let* $\phi_0 = (P_0,n_0) \in C$, *then there exists a unique solution* $\phi(t) = (P(t),\ n(t)) \in C$ *of the system (4.1) - (4.5).*

One way to obtain this result is to apply standard local existence and uniqueness results for retarded functional differential equations to the integro-differential equation obtained above (see Hale (1977)). As in section 2 of this chapter, global existence follows if one can give an a priori estimate on the solution: see lemma 4.10 below.

Now we can define a nonlinear semigroup (or dynamical system) $T(t)$ on C in the following standard way:

$$\phi(t) = T(t)\phi_0,\ t \geqslant 0,\ \phi_0 \in C\ .$$

It is incorporated in the definition of solutions that $T(t)$ is nonnegativity-preserving. However, it is an easy task to show that $\phi_0 \in C$, $\phi_0 \neq 0$, implies that $\phi(t) = T(t)\phi_0 >> 0$ for t large enough. This fact will be exploited in the proof of theorem 4.27.

4.3. *Boundedness of solutions*

We conclude from (4.8) that it suffices to show that $P(t)$ remains bounded for all $t>0$.

First we integrate (4.2) from 0 to 1. Let

$$N(t) = \int_0^1 n(t,x)dx \ ,$$

then

$$\frac{dN}{dt}(t) = -\mu N(t) + \gamma(P(t))P(t) - \int_0^1 b(x)n(t,x)dx \ .$$

EXERCISE 4.9. Check this.

Combining this with (4.1) yields

$$\frac{dM}{dt} = -\mu M + \gamma(P)P \ , \tag{4.11}$$

where $M(t)=2N(t) + P(t)$, $t \geqslant 0$.

LEMMA 4.10. *$M(t)$ is bounded, $t \geqslant 0$.*

PROOF. Suppose that $M(t)$ is not bounded. Then we can choose a strictly increasing sequence $t_n>0$, $n \in \mathbb{N}$ such that $M(t_n) \to \infty$ if $n \to \infty$ and $\dot{M}(t_n) \geqslant 0$.

(i) Suppose that $P(t_n)$ is a bounded sequence, $P(t_n) \leqslant P_{\max}$, $n \in \mathbb{N}$, then $\dot{M}(t_n) \leqslant \gamma(0)P_{\max} - \mu M(t_n)$, and therefore $\dot{M}(t_n)<0$ if n is large enough which is a contradiction.

(ii) Suppose that $P(t_n) \to \infty$ as $n \to \infty$. Then $\gamma(P(t_n)) < \frac{1}{2}\mu$ as n is large enough. Since $\dot{M}(t_n) \leqslant (\gamma(P(t_n)) - \mu)P(t_n)$ this contradicts $\dot{M}(t_n) \geqslant 0$. □

Since $0 \leqslant P(t) \leqslant M(t)$, lemma 4.10 implies boundedness of $P(t)$ and hence of the solutions of (4.1) - (4.5).

THEOREM 4.11. *Every solution* $(P(t), n(t))$ of (4.1) - (4.5) *is bounded and precompact.*

EXERCISE 4.12. Show that precompactness follows from (4.8).

4.4. *Extinction of the population*

It is intuitively clear that, if the population even under the most favourable growth conditions (i.e. $\gamma(P(t)) = \gamma(0)$, for all $t \geqslant 0$) does become extinct, then there is no hope for survival under the actual circumstances. Below we shall make this intuitive idea more precise.

EXERCISE 4.13. Use Exercise 4.4 to show that for every fixed $\gamma \geqslant 0$ there exists a $\mu(\gamma) \geqslant 0$ such that for $\mu \geqslant 0$ the dominant eigenvalue λ_d is given by $\lambda_d = \mu(\gamma) - \mu$. Show that $\mu(\gamma)$ is increasing with γ.
Hint: The characteristic equation can be rewritten as

$$\lambda + \mu + \gamma = 2\gamma \int_0^1 k_0(x) e^{-(\lambda+\mu)G(x)}dx \ ,\text{where } \int_0^1 k_0(x)dx = 1 \ .$$

Let for $\gamma \geqslant 0$ and $\lambda \in \mathbb{C}$

$$\pi_\gamma(\lambda) = \frac{2\gamma}{\lambda+\mu+\gamma}\int_0^1 k(x)e^{-\lambda G(x)}dx\ .$$

EXERCISE 4.14. Compare this expression to the characteristic equation obtained in Exercise 4.4. Interpret $\pi_\gamma(0)$ as the net reproduction rate (i.e. the average number of offspring of a newborn individual) if the transition probability is γ.

Now we can state our "extinction result".

THEOREM 4.15. *If* $\pi_{\gamma(0)}(0)\leqslant 1$, *then the population becomes extinct, i.e. the solution* $(P(t),\ n(t))$ *of* (4.1) - (4.5) *satisfies*

$$(P(t),\ n(t))\to 0 \text{ as } t\to\infty\ .$$

EXERCISE 4.16. Show that the condition in this theorem is satisfied iff $\mu\geqslant\mu(\gamma(0))$ (see Exercise 4.13).

In order to prove theorem 4.15 we shall construct a suitable Lyapunov function (see section 1 of this chapter). Let

$$r(x) = \frac{2}{E(x)}\int_x^1 k(y)dy\ . \tag{4.12}$$

EXERCISE 4.17. Interpret $r(x)$ as the expected number of offspring of a cell in the M-phase with maturity x. Show that $r(x) = 2$ if $\mu = 0$.

We define the continuous function $\mathcal{V}$ on X by:

$$\mathcal{V}(\rho,\nu) = \rho + \int_0^1 r(x)\nu(x)dx,\ (\rho,\nu)\in X\ . \tag{4.13}$$

We can give the following intuitive interpretation of the function $\mathcal{V}$. Obviously an individual in the M-phase has a greater chance to divide eventually than an individual in the G-phase. Since we can interpret $r(x)$ as the expected number of offspring of an individual in the M-phase this function assigns a value to every individual, representing its (expected) future contribution to the population.

Let $(P_0,n_0)\in C$ and $(P(t),\ n(t)) = T(t)(P_0,n_0),\ t\geqslant 0$, then

$$\frac{d}{dt}\mathcal{V}(P(t),n(t)) = ((r(0)-1)\gamma(P(t))-\mu)P(t)\ .$$

EXERCISE 4.18. Use $r'(x) = \dfrac{\mu+b(x)}{V(x)}r(x) - \dfrac{2b(x)}{V(x)}$ to prove this.

Hence $\dot{\mathcal{V}}(\rho,\nu) = ((r(0)-1)\gamma(\rho)-\mu)\rho,\ (\rho,\nu)\in C$.

EXERCISE 4.19. Prove that $\dot{\mathcal{V}}(\rho,\nu)\leqslant 0$, for all $(\rho,\nu)\in C$ iff $\pi_{\gamma(0)}(0)\leqslant 1$.

As in section 1 of this chapter we define

$$E = \{(\rho,\nu)\in C \mid \dot{\mathcal{V}}(\rho,\nu) = 0\}\ ,$$

and we let $\mathfrak{E}$ be the largest invariant subset of E. Suppose that $\pi_{\gamma(0)}(0)\leqslant 1$. Then

$$E = \{(\rho,\nu)\in C \mid \rho = 0\}\ .$$

Let $(\rho,\nu)\in\mathfrak{E}$. Then $\rho = 0$. Since $\mathfrak{E}$ is invariant it follows from (4.8) that $\nu = 0$, hence

$$\mathfrak{E} = \{(0,0)\}\ .$$

Since moreover, for every $(P_0,n_0)\in C$ the orbit $\Gamma^+(P_0,n_0)$ is precompact (theorem 4.11) we obtain from the invariance principle (theorem 1.1.6) that $(P(t),n(t))\to(0,0)$ as $t\to 0$, and this proves theorem 4.15.

4.5. *Existence of a nontrivial equilibrium and monotonicity on an invariant subset*

It follows from theorem 4.15 and exercise 4.16 that the trivial equilibrium is (globally) stable if the deathrate μ is large enough.

EXERCISE 4.20. Show that there exists a unique nontrivial equilibrium $(\hat{P},\hat{n})$ of (4.1) - (4.3) if $\pi_{\gamma(0)}(0)>1$, where $\hat{n}(x) = f(\hat{P})\frac{E(x)}{V(x)}$, and $\hat{P}$ is determined by

$$\gamma(\hat{P}) = \frac{\mu}{r(0)-1} \quad (\text{i.e } \pi_{\gamma(\hat{P})}(0) = 1).$$

(Recall the corresponding result for the Daphnia-model in section I.3.)

We assume for the rest of this section that $\pi_{\gamma(0)}(0)>1$. This is equivalent to

$$\frac{r(0)\gamma(0)}{\mu+\gamma(0)} >1 \tag{4.14}$$

From assumption 4.5 we conclude that $f(P) = P\gamma(P)$ is increasing for small values of P. From a biological point of view the following assumption means no restriction of generality.

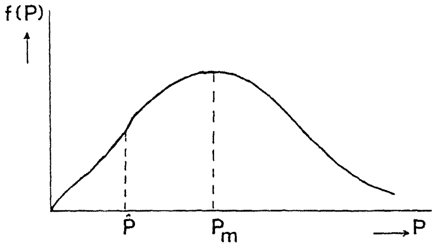
ASSUMPTION 4.21. There exists a P_m, $0<P_m\leqslant\infty$ such that f is increasing on $[0,P_m)$ and nonincreasing on (P_m,∞).

We recall that γ is decreasing. We also make the following

ASSUMPTION 4.22. $\hat{P}<P_m$.

For future use we note that assumption 4.22 can be reformulated as

$$(r(0)-1)\gamma(P_m)<\mu\,. \tag{4.15}$$



For $P\geqslant 0$ we define $\Phi_P\in C$ by

$$\Phi_P = (P,f(P)\frac{E(\cdot)}{V(\cdot)})\,. \tag{4.16}$$

Let $\hat{P}<\tilde{P}<P_m$ and let the bounded subset $\tilde{C}\subseteq C$ be given by

$$\tilde{C} = \{\phi\in C \mid \phi<<\Phi_{\tilde{P}}\}\,. \tag{4.17}$$

LEMMA 4.23. $\tilde{C}$ is positively invariant under the action of $T(t)$.

PROOF. Let $\phi_0\in\tilde{C}$ and suppose that $T(t)\phi_0\notin\tilde{C}$ for some $t>0$. Let t_0 be the smallest t for which this is so, and let $T(t)\phi_0 = (P(t), n(t))$. There are three possibilities:
i) There exists an $x, 0<x\leqslant 1$, such that $\frac{V(x)}{E(x)}n(t_0,x) = f(\tilde{P})$. Let $h>0$ be such that $t-h>0$ and $X(-h,x)>0$, then $\frac{V(X(-h,x))}{E(X(-h,x))}n(t_0-h,X(-h,x))= \frac{V(x)}{E(x)}n(t_0,x)=f(\tilde{P})$ as follows directly by integration of (4.2) along characteristics. But this yields $T(t_0-h)\phi\notin\tilde{C}$ which is contradictory with the definition of t_0.
ii) Let $V(0)n(t_0,0) = f(\tilde{P})$. Then $f(P(t_0)) = f(\tilde{P})$ and this implies that $P(t_0) = \tilde{P}$.
ii) The third possibility is $P(t_0) = \tilde{P}$.
Therefore we may assume $P(t_0) = \tilde{P}$. Since $P(t_0-h)<\tilde{P}$ for $0<h\leqslant t_0$ we find that $\dot{P}(t_0)\geqslant 0$. On the other hand

$$\dot{P}(t_0) = -\mu\tilde{P} - f(\tilde{P}) + 2\int_0^1 b(x)n(t_0,x)dx < -\mu\tilde{P} - f(\tilde{P}) + 2\int_0^1 k(x)f(\tilde{P})dx = (r(0)-1)f(\tilde{P}) - \mu\tilde{P} < 0$$

because $\tilde{P} > \hat{P}$. Thus we have obtained a contradiction and the result is proved. □

In the sequel we shall need the following technical lemma.

LEMMA 4.24.

a)*Let* $\phi \in C$, $\phi >> 0$, *then there exists a sequence* $\{\phi_k\}_{k \in \mathbb{N}}$ in C *such that* $0 << \phi_k << \phi$, $k \in \mathbb{N}$ *and* $\lim_{k \to \infty} \phi_k = \phi$.

b)*Let* $\phi \in \tilde{C}$, *then there exists a sequence* $\{\phi_k\}_{k \in \mathbb{N}}$ *in* $\tilde{C}$ *such that* $\phi_k >> \phi$, $k \in \mathbb{N}$ *and* $\lim_{k \to \infty} \phi_k = \phi$.

PROOF. We shall only give a proof of b).
Let $\phi = (\rho,\nu) \in \tilde{C}$. Then $\rho < \tilde{P}$ and $\frac{V(x)}{E(x)}\nu(x) < f(\tilde{P})$, $0 \leq x \leq 1$. Let $\{a_k\}_{k \in \mathbb{N}}$ be a sequence in $\mathbb{R}$ such that $a_k \to 0$, $k \to \infty$ and for all $k \in \mathbb{N}$: $a_k > 0$ and $\frac{V(x)}{E(x)}\nu(x) + a_k < f(\tilde{P})$, $0 \leq x \leq 1$. Let $\nu_k(x) = \nu(x) + a_k \frac{E(x)}{V(x)}$, and let $\rho_k \in (\rho, \tilde{P})$ be determined by $f(\rho_k) = V(0)\nu_k(0) = V(0)\nu(0) + a_k = f(\rho) + a_k$. Then $\phi_k = (\rho_k, \nu_k)$, $k \in \mathbb{N}$, satisfies the conditions of the lemma. □

We shall now prove two monotonicity results.

THEOREM 4.25. $T(t)$ *is monotone on* $\tilde{C}$, *i.e.* $\phi, \psi \in \tilde{C}$ *and* $\phi \leq \psi$ *imply that* $T(t)\phi \leq T(t)\psi$, $t \geq 0$.

PROOF. Let $\phi, \psi \in \tilde{C}$, $\phi \leq \psi$ and let $\{\psi_k\}_{k \in \mathbb{N}}$ be a sequence in $\tilde{C}$ such that $\psi_k \to \psi$, $k \to \infty$ and $\psi_k >> \psi$, $k \in \mathbb{N}$ (cf. lemma 4.24b). We show that $T(t)\phi << T(t)\psi_k$ for all $t > 0$ and $k \in \mathbb{N}$. Suppose there is a $k \in \mathbb{N}$ for which this is not true, and let $t_0 > 0$ be the smallest t for which the strict inequality is not satisfied. Let $T(t)\phi = (P(t;\phi), n(t;\phi))$ and $T(t)\psi_k = (P(t;\psi_k), n(t;\psi_k))$. As in the proof of lemma 4.23 we can show that $P(t_0;\phi) = P(t_0;\psi_k)$. Since $P(t_0 - h;\phi) < P(t_0 - h;\psi_k)$, $0 < h \leq t_0$ we conclude that

$$\dot{P}(t_0;\phi) \geq \dot{P}(t_0;\psi_k) .$$

This, however, implies that

$$2\int_0^1 b(x)n(t_0,x;\phi)dx \geq 2\int_0^1 b(x)n(t_0,x;\psi_k)dx ,$$

which is a contradiction. Now let $t > 0$ be fixed. Then $T(t)\phi << T(t)\psi_k$, $k \in \mathbb{N}$. Letting $k \to \infty$ and using the continuity of $T(t)$ we find

$$T(t)\phi \leq T(t)\psi$$

and the result follows. □

THEOREM 4.26.

a) *If* $0 < P < \hat{P}$ *then* $T(t)\Phi_P$ *is increasing in t.*

b) *If* $\hat{P} < P < \tilde{P}$ *then* $T(t)\Phi_P$ *is decreasing in t.*

PROOF. We shall only prove a). The proof of b) proceeds along the same lines. Let $0 < P < \hat{P}$, and let Q be such that $P < Q < \hat{P}$. Suppose we can show that

$$\Phi_P << T(t)\Phi_Q, \; t > 0 .$$

Then, letting Q approach P, we obtain

$$\Phi_P \leq T(t)\Phi_P, \; t > 0 ,$$

and now the monotonicity of $T(t)$ gives

$$T(s)\Phi_P \leq T(s)T(t)\Phi_P = T(s+t)\Phi_P, \; s \geq 0, \; t \geq 0$$

which would imply the result. Therefore we shall prove that indeed $\Phi_P << T(t)\Phi_Q$ for all $t > 0$. Suppose not. Again let t_0 be the smallest t such that the strict inequality is not satisfied. As in the proof of lemma 4.23 we can show that

$$P = P(t_0;\Phi_Q)\,.$$

Here $T(t)\Phi_Q = (P(t;\Phi_Q),\ n(t;\Phi_Q))$. Since $P<P(t;\Phi_Q)$, $0\leq t<t_0$, we obtain

$$\dot{P}(t_0;\Phi_Q)\leq 0\,.$$

On the other hand

$$\dot{P}(t_0;\Phi_Q) = -\mu P-f(P) + 2\int_0^1 b(x)n(t_0,x,\Phi_Q)dx > -\mu P-f(P) + 2\int_0^1 b(x)f(P)\frac{E(x)}{V(x)}dx$$
$$= -\mu P-f(P) + r(0)f(P)>0\,,$$

since $P<\hat{P}$. This is a contradiction, and the result is proved. □

4.6. *Global stability of the nontrivial equilibrium*

In this section we make again the assumptions 4.21 and 4.22. Let $\tilde{P}$ satisfy $\hat{P}<\tilde{P}<P_m$ and let the invariant bounded subset $\tilde{C}$ be given by (4.17). Suppose that the initial condition $\phi_0\neq 0$ is contained in $\tilde{C}$. From the remark following theorem 4.8 we obtain that there exists a $t_1>0$ such that

$$T(t_1)\phi_0 \gg 0\,.$$

A straightforward calculation shows that there exist $\underline{P},\bar{P}$ such that

$$0<\underline{P}\leq\hat{P}\leq\bar{P}<\tilde{P} \text{ and } \Phi_{\underline{P}}\leq T(t_1)\phi_0\leq\Phi_{\bar{P}}\,.$$

Since $\{T(t)\Phi_{\underline{P}}\}_{t\geq 0}$ and $\{T(t)\Phi_{\bar{P}}\}_{t\geq 0}$ define a precompact increasing and decreasing net respectively, we may conclude that both nets converge to a limit which defines a fixed point of $T(t)$. But the only fixed point is $\phi = (\hat{P},\hat{n})$ (see exercise 4.20) and therefore

$$\lim_{t\to\infty} T(t)\Phi_{\underline{P}} = \lim_{t\to\infty} T(t)\Phi_{\bar{P}} = \hat{\phi}\,.$$

We conclude from

$$T(t-t_1)\Phi_{\underline{P}}\leq T(t)\phi_0\leq T(t-t_1)\Phi_{\bar{P}}$$

that

$$\lim_{t\to\infty} T(t)\phi_0 = \hat{\phi}\,,$$

and we have proved the following result

THEOREM 4.27. *Let* $\phi_0\in\tilde{C}\setminus\{0\}$, *then* $\lim\limits_{t\to\infty} T(t)\phi_0 = \hat{\phi}$.

We can prove our main result now.

THEOREM 4.28. *Let* $\phi_0\in C$, $\phi_0\neq 0$, *then* $\lim\limits_{t\to\infty} T(t)\phi_0 = \hat{\phi}$.

PROOF. i) Suppose $P_m = \infty$ Let $\phi_0\in C$, $\phi_0\neq 0$. If $f(P)\to\infty$ as $P\to\infty$ then the proof follows from the fact that $\phi_0\in\tilde{C}$ if $\tilde{P}$ is large enough. If $f(P)\to f_\infty<\infty$ as $P\to\infty$, then we conclude from (4.8) that for $t>G(1)$ we have $T(t)\phi_0\in\tilde{C}$ if $\tilde{P}$ is large enough.
ii) Let $P_m<\infty$. Let $\phi_0\in C$, $\phi_0\neq 0$, and $(P(t),n(t)) = T(t)\phi_0$. Suppose $P(t)\geq P_m$ for all $t\geq t_0$ where $t_0>0$. Now let $(\rho,\nu)\in\Omega(\phi_0)$ (i.e. the omega limit set of ϕ_0, cf. section 1) then $\rho\geq P_m$, and $\dot{V}(\rho,\nu) = (r(0)-1)f(\rho)-\mu\rho<0$ which is impossible.

We may conclude that there exists a $t_1\geq G(1)$ such that $P(t_1)<P_m$. Let $\tilde{P}$, $P(t_1)<\tilde{P}<P_m$ be such that

$$-\mu\hat{P}-f(\tilde{P}) + r(0)f(P_m)<0$$

(note that such a $\tilde{P}$ exists since $-\mu P_m + (r(0)-1)f(P_m)<0$). We show that $P(t)<\tilde{P}$ for all $t\geq t_1$. Suppose not. Let t_2 be the smallest value of t greater than t_1 such that $P(t_2) = \tilde{P}$. Then $\dot{P}(t_2)\geq 0$. On the other hand

$$\dot{P}(t_2) = -\mu\tilde{P}-f(\tilde{P}) + 2\int_0^1 b(x)n(t_2,x)dx \leq -\mu\tilde{P}-f(\tilde{P}) + 2\int_0^1 k(x)f(P_m)dx = -\mu\tilde{P}-f(\tilde{P}) + r(0)f(P_m)<0\,,$$

which is a contradiction. Therefore

$$P(t)<\tilde{P},\ t\geq t_1\ ,$$

and from (4.8) we conclude that

$$T(t)\phi_0 = (P(t),n(t))\in\tilde{C},\ t\geq t_1 + G(1)\ ,$$

where $\tilde{C}$ is given by (4.17). This proves the result. □

4.7. *Final remarks*

Kirk, Orr and Forrest (1970) present a model describing the control of the bone marrow stem cell population, which supplies the circulating blood population. They assumed that the production process was controlled by some stem cell specific mitotic inhibitor (one of the family of chalones), and that the mitotic phase was of constant duration. In terms of our G-M-model this last assumption is equivalent to the supposition that all cells divide at reaching maturity x = 1. Using our notation their model is described by

$$\frac{dP}{dt} = -\mu P - \tilde{\gamma}(c)P + 2V(1)n(t,1)\ , \tag{4.18}$$

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x}(V(x)n(t,x)) = -\delta n(t,x)\ , \tag{4.19}$$

$$V(0)n(t,0) = \tilde{\gamma}(c)P\ , \tag{4.20}$$

$$\frac{dc}{dt} = \tilde{\rho}P - \tilde{\sigma}c\ . \tag{4.21}$$

Here c denotes the concentration of the mitotic inhibitor produced by G-cells at a rate $\tilde{\rho}$, and desintegrating at a rate $\tilde{\sigma}$. μP is due to loss from the G-phase via differentiation into the various channels. Kirk, Orr and Forrest (1970) assumed $\delta = 0$ (Mackey (1978) calls this the 'normal' situation) and they 'solved' the system by using analogue computer techniques. This model was also studied by Mackey (1978, 1981) and he calls it the *pluripotential stem cell model.* Mackey extensively examines the case that the dynamics of c is much faster than the dynamics of P and n, which can be modelled by assuming that $\tilde{\rho} = \rho/\epsilon$, $\tilde{\sigma} = \sigma/\epsilon$,, where ϵ is a small parameter. Keeping ρ and σ constant and letting ϵ tend to zero (4.21) can be replaced by

$$\rho P = \sigma c$$

(the so-called quasi-steady state situation). Now we may substitute in (4.18) and (4.20)

$$\tilde{\gamma}(c) = \tilde{\gamma}(\frac{\rho}{\sigma}P) = \gamma(P)\ ,$$

which happens to be the case that we considered. Additionally Mackey assumes that the transition probability γ depends on the total G-population in a decreasing manner, to be precise

$$\gamma(P) = \frac{\gamma_0\theta^q}{\theta^q + P^q}$$

for some integer q and positive constants γ_0 and θ.

Exercise 4.29. Assume the quasi-steady state situation (i.e. $\tilde{\gamma}(c) = \gamma(P)$) and let τ be the duration of the M-phase (i.e. $\tau = G(1)$). Let $P(t),n(t,x)$ be a solution of the system thus obtained and $N(t) = \int_0^1 n(t,x)$. Show that $P(t)$, $N(t)$ obey the delay-differential equations.

$$\frac{dP}{dt} = -\mu P - \gamma(P)P + 2\gamma(P_\tau)P_\tau e^{-\delta\tau}\ ,\ t>\tau\ , \tag{4.22}$$

$$\frac{dN}{dt} = -\delta N + \gamma(P)P - \gamma(P_\tau)P_\tau e^{-\delta\tau}\ ,\ t>\tau\ , \tag{4.23}$$

where $P_\tau(t) = P(t-\tau)$ and $N_\tau(t) = N(t-\tau)$.

It is easily seen that under some conditions on the parameters a unique non-trivial equilibrium $\hat{P}$, $\hat{N}$ of (4.22) - (4.23) exists. Numerical experiments suggest local stability of this equilibrium in some subset of the parameter space;

these experiments also indicate the occurrence of Hopf bifurcations (and ultimately the existence of solutions of a more 'complex' nature). As a matter of fact this local stability and the onset of Hopf bifurcation can be determined from the characteristic equation which can be obtained by linearizing around the equilibrium $\hat{P}$.

Exercise 4.30. Convince yourself that this characteristic equation takes the form

$$\lambda + A + Be^{-\lambda\tau} = 0 ,$$

where A,B are constants depending on the parameters.

The situation becomes completely different (as well in our $G-M$-model as in the variant described above) if the growthrate V depends on time, caused by environmental factors. Let us, as a simple example, consider (4.18) - (4.20) where $\tilde{\gamma}(c)$ is assumed to be a constant γ and where the growthrate V depends only on P (and not on x).

$$\frac{dP}{dt} = -\mu P - \gamma P + 2V(P)n(t,1) \tag{4.24}$$

$$\frac{\partial n}{\partial t} + V(P)\frac{\partial n}{\partial x} = -\delta n(t,x) \tag{4.25}$$

$$V(P)n(t,0) = \gamma P . \tag{4.26}$$

This system cannot be reduced to a two-dimensional system of delay-differential equations with fixed delay, because the duration of the M-phase depends on the moment of entry. Let us define $\tau(t)$ by

$$1 = \int_{t-\tau(t)}^{t} V(P(s))ds .$$

Exercise 4.31. Interpret $\tau(t)$ as the time which an individual reaching maturity 1 at time t has spent in the M-phase.

We can reduce (4.24) - (4.26) to the system

$$\frac{dP}{dt}(t) = -(\mu+\gamma)P(t) + 2\gamma e^{-\delta\tau(t)}P(t-\tau(t))$$

$$\frac{dN}{dt}(t) = -\delta N(t) + \gamma P(t) - \gamma e^{-\delta\tau(t)}P(t-\tau(t)) ,$$

$$\frac{d\tau}{dt}(t) = 1 - \frac{V(P(t))}{V(P(t-\tau(t))} ,$$

where $N(t) = \int_0^1 n(t,x)dx$, i.e. a three-dimensional system of variable-delay differential equations, and such a reduction has been extensively investigated by Nisbet and Gurney (1983). See also the contribution by Gurney, Nisbet and Blythe to part B of these lecture notes.

Exercise 4.32. Show that the above reduction to the three-dimensional system of variable delay-differential equations is indeed correct.

5. Reduction to an ODE-system: a chemostat model for a cell population reproducing by unequal fission

In this section we consider a model for the growth of a cell population living in a chemostat (cf. section I.4) and reproducing by unequal fission. The main feature of the model is that it can be reduced to an ODE-system in the sense that the dynamics of substrate and biomass (represented by the total population) are governed by the classical two dimensional chemostat system (cf. Herbert et al. (1956), Waltman (1983), and section I.4). The knowledge of the behaviour of solutions of this system makes it possible to characterize the omega limit sets of solutions of the full problem.

5.1. The model

As in section I.4 and chapter II it is assumed that a cell is fully characterized by its size x. However in the present model fission not necessarily results in equal parts, but the ratio p between the birth size of a daughter and the division size of her mother is a random variable described by a smooth probability density function $d(p)$, which does not depend on the division size of the mother: since size is conserved at division, d is symmetric around $p = \frac{1}{2}$. Moreover

$$\int_0^1 d(p)dp = 1 .$$

For more biological details we refer to Koch and Schaechter (1962).

Now we shall first write down the equations:

$$\frac{\partial n}{\partial t}(t,x) + \frac{\partial}{\partial x}(\beta(S(t))xn(t,x)) = -Dn(t,x)-b(x)n(t,x) + 2\int_0^1 \frac{d(p)}{p}b(\frac{x}{p})n(t,\frac{x}{p})dp , \tag{5.1}$$

$$\frac{dS}{dt}(t) = D(S^i-S(t))-\alpha\beta(S(t))\int_{x_{\min}}^1 xn(t,x)dx . \tag{5.2}$$

See section III.3.3 for a derivation of the corresponding linear equation (see also Remark I.4.3.4, Exercise I.4.3.6 and Exercise II.11.3). Here $n(t,x)$ denotes the size distribution (per unit of volume), $S(t)$ is the concentration of the limiting nutrient in the chemostat (see section I.4.5 for a description of the chemostat device). D is the dilution rate, $b(x)$ is the division rate, S^i is the concentration of the limiting substrate in the inflowing fluid, α is a conversion factor describing the relation between size-units and substrate units ($1/\alpha$ is sometimes called the yield constant) and $x_{\min}$ is the minimum size. The growth rate V of the individuals depends on S, but is at every fixed time proportional to the individual's size

$$V = \beta(S(t))x .$$

Among others this means that we assume the 'Structural Nutrient Hypothesis': see DIEKMANN, LAUWERIER, ALDENBERG & METZ (1983) and § I.4.5. We make the following assumptions on β, d and b.

ASSUMPTIONS 5.1.

H_β: $\beta(S) = \dfrac{k_1 S}{1+k_2 S}$, where $k_1,k_2>0$.

H_d: $d(p)>0$, $p\in(\frac{1}{2}-\Delta, \frac{1}{2}+\Delta)$ and $d(p) = 0$ outside this interval. $d(p)$ is symmetric around $p = \frac{1}{2}$, $\int_0^1 d(p)dp = 1$ and d is continuously differentiable on $[\frac{1}{2}-\Delta, \frac{1}{2}+\Delta]$.

H_b:b is continuous on $[0,1)$, $b(x)>0$, $x\in(a,1)$ and $b(x) = 0, x\leqslant a$. Moreover $\lim_{x\uparrow 1}\int_a^x b(\xi)d\xi = \infty$, and the function $x\rightarrow\dfrac{b(x)}{\bar\beta x}\exp[-\int_a^x \dfrac{b(\xi)}{\bar\beta\xi}d\xi]$ is bounded, where $\bar\beta = \lim_{S\to\infty}\beta(S) = \dfrac{k_1}{k_2}$.

The function β in H_β is called the Monod-Michaelis-Menten function (cf. Remark I.4.5.2). H_β is only made for simplicity: essential is that $\beta(0) = 0$, β is increasing and $\lim_{S\to\infty}\beta(S)$ exists and is finite.

The last condition in H_b says that the function $\phi_b(x)$ given by (I.4.1.5) for $V(x) = \bar\beta x$ is bounded.
Note that it follows from these assumptions that

$$x_{\min} = a\,(\tfrac{1}{2}-\Delta) ,$$

and we have to supply (5.1) - (5.2) with the boundary condition

$$n(t,x_{\min}) = 0 . \tag{5.3}$$

dditionally we impose the initial conditions

$$n(0,x) = n_0(x)\geqslant 0 \tag{5.4}$$

$$S(0) = S_0\geqslant 0 . \tag{5.5}$$

In this section we shall try to avoid as many technical details as possible: an exhaustive investigation of the problem can be found in Heijmans (1984b).

5.2. *The linear equation*

It appears sensible to start with an investigation of the linear problem that is obtained if we assume that β does not depend on $S(t)$, but is a constant. The analysis of this linear problem proceeds along the lines of chapter II, and this we consider a justification for omitting the technical details. Let

$$E(x) = \exp(-\int_a^x \frac{b(\xi)}{\beta\xi}\, d\xi)\,.$$

If we assume

$H_{0,\text{linear}}$: $n_0(\cdot)\,/\,E(\cdot)$ belongs to $L^1[x_{\min},1]$,

(a very similar condition is imposed in chapter II, the main difference being that we considered continuous functions at that place) then we can prove that there exists a unique solution $n(t,x)$ of

$$\frac{\partial n}{\partial t}(t,x) + \frac{\partial}{\partial x}(\beta x n(t,x)) = -Dn(t,x) - b(x)n(t,x) + 2\int_{\frac{1}{2}-\Delta}^{\frac{1}{2}+\Delta} \frac{d(p)}{p}\, b(\frac{x}{p})n(t,\frac{x}{p})dp \tag{5.6}$$

$$n(t,x_{\min}) = 0 \tag{5.7}$$

$$n(0,x) = n_0(x)\,. \tag{5.8}$$

We can associate a strongly continuous linear semigroup T_β with the problem in the following way

$$T_\beta(t)n_0 = n(t),$$

where $n(t) = n(t,.)$ is the solution of (5.6) - (5.8) and the subindex β indicates the dependence on β. For a function $\phi \in L^1[x_{\min},1]$ we define

$$W[\phi] = \int_{x_{\min}}^{1} x\phi(x)dx\,. \tag{5.9}$$

This quantity can be interpreted as the biomass of a population whose size distribution is described by ϕ, and it plays a major role in the subsequent analysis.

EXERCISE 5.2. Let n_0 obey $H_{0,\text{linear}}$ and let $n(t) = T_\beta(t)n_0$, then $W(t) \overset{def}{=} W[n(t)]$ obeys the ordinary differential equation:

$$\frac{dW}{dt} = (\beta - D)W(t),\ W(0) = W[n_0].$$

Prove this.

It can be shown rigorously that the infinitesimal generator associated with (5.6) - (5.7) has a strictly dominant eigenvalue λ_d,

$$\lambda_d = \beta - D,$$

which is a pole of the resolvent and algebraically simple. The corresponding spectral projection is given by

$$P\phi = W[\phi]\cdot n_d\,, \tag{5.10}$$

where n_d is the positive eigenvector of A belonging to eigenvalue λ_d normalized by $W[n_d] = 1$. n_d is called the stable size distribution. Sometimes we shall write $n_d(\beta)$ to indicate the dependence on β.

EXERCISE 5.3. See whether you can find a relation between this result and the outcome of Exercise 5.2!

Now the following result holds:

THEOREM 5.4. *There exist constants* $\epsilon, M > 0$ *such that*

$$\|T_\beta(t)n_0 - W[n_0]e^{(\beta-D)t}n_d\| \leqslant Me^{(\beta-D-\epsilon)t}\|n_0\|,$$

for all $t \geqslant 0$ *and initial functions satisfying* $H_{0,\text{linear}}$.

Note that there exists a stable size distribution even though $V(2x) = 2V(x)$ for all x. This, of course, is due to the fact that d is not a delta function but smooth.

5.3. An O.D.E. system related to the nonlinear problem.

If we multiply (5.1) by x and integrate (as in Exercise 5.2) we obtain

$$\frac{dW}{dt}(t) = (\beta(S(t)) - D)W(t) \tag{5.11}$$

where now $W(t) = W(t;S_0, n_0) = W[n(t;S_0,n_0)]$, if a solution $S(t;S_0,n_0)$, $n(t;S_0,n_0)$ of (5.1) - (5.5) exists. Furthermore we can rewrite (5.2) as

$$\frac{dS}{dt}(t) = D(S^i - S(t)) - \alpha\beta(S(t))W(t) , \tag{5.12}$$

and we have obtained a two-dimensional O.D.E. system, called the Monod equations which has been extensively investigated in the literature (e.g. Herbert et al. (1956), Hsu et al. (1977)). For the following results we refer to this last reference. The system (5.11) - (5.12) supplied with $W(0) = W_0 \geq 0, S(0) = S_0 \geq 0$ has a unique bounded solution. Obviously (5.11) - (5.12) always has the trivial equilibrium $W = 0, S = S^i$. Let

$$D_{crit} = \frac{k_1 S^i}{1 + k_2 S^i}$$

where k_1, k_2 are defined by H_β.

Exercise 5.5. Show that there exists a unique nontrivial equilibrium $\hat{S} = \frac{D}{k_1 - k_2 D}$, $\hat{W} = \frac{1}{\alpha}(S^i - \frac{D}{k_1 - k_2 D})$ if $D < D_{crit}$.

Moreover it can be proved that this equilibrium is globally asymptotically stable if $D < D_{crit}$.

Exercise 5.6. Let $D < D_{crit}$. Let for $S, W > 0$, $\mathcal{V}$ be defined as:

$$\mathcal{V}(S,W) = (S - \hat{S} - \hat{S}\log\frac{S}{\hat{S}}) + c(W - \hat{W} - \hat{W}\log\frac{W}{\hat{W}}) ,$$

where

$$c = \alpha k / (k_1 - k_2 D).$$

Show that $\dot{\mathcal{V}}(S,W) = -\frac{(S-\hat{S})^2}{S(1+k_2 S)} \; (-(k_1 - k_2 D)S^i - k_2 DS) \leq 0$ and that global stability of $\hat{S}$, $\hat{W}$ follows from the invariance principle (theorem 1.6).

5.4. The nonlinear problem.

Now let us return to our original nonlinear problem (5.1) - (5.5). The observation that the nonlinear function $\beta(S(\cdot))$ can be defined a priori as a function of time t makes the existence and uniqueness proof a relatively easy one. More precisely: it can be shown that for every initial pair (S_0, n_0) satisfying $S_0 \geq 0$, $n_0(x) \geq 0$ and some condition $H_{0,\text{nonlinear}}$, which we shall not write down explicitly, but which is the nonlinear analogue of $H_{0,\text{linear}}$, there exists a unique solution which we denote with $(S(t;S_0,n_0), n(t;S_0,n_0))$, or briefly $(S(t), n(t))$.

Before we proceed we shall give some definitions. Let X be the space of pairs (σ,ν) with norm $\|(\sigma,\nu)\|_X = |\sigma| + \|\nu\|$. Let C be the subset of X consisting of all pairs (σ,ν) such that $\sigma \geq 0$, $\nu \geq 0$ and (σ,ν) satisfies $H_{0,\text{nonlinear}}$.

Exercise 5.7. If $D \geq D_{crit}$ then the only equilibrium is the trivial equilibrium $(S^i, 0)$. If $D < D_{crit}$ then there exists a unique nontrivial equilibrium $(\hat{S}, \hat{n})$ where $\hat{n} = \hat{W} n_d(D)$ and $\hat{S}$, $\hat{W}$ are determined by Exercise 5.5.

Now we can state the main result of this section.

THEOREM 5.8. *Let* $(S_0,n_0)\in C$, $(S_0,n_0)\neq(0,0)$ *and let* $(S(t),n(t))$ *be the associated solution of* (5.1) - (5.5). *If* $D\geqslant D_{crit}$ *then* $\lim_{t\to\infty} S(t) = S^i$, $\lim_{t\to\infty} n(t) = 0$. *If* $D<D_{crit}$ *then* $\lim_{t\to\infty} S(t) = \hat{S}$, $\lim_{t\to\infty} n(t) = \hat{n}$.

The remainder of this section is concerned with a sketch of the proof of this global stability result. The missing details can be found in Heijmans (1984).

Let $D<D_{crit}$. We can associate a dynamical system with the problem in the standard way: $T(t)(S_0,n_0) = (S(t;S_0,n_0), n(t;S_0,n_0))$, for $(S_0,n_0)\in C$. Now $T(t)$ leaves C invariant. Let $(S_0,n_0)\in C\setminus\{(0,0)\}$ and let $\Gamma^+(S_0,n_0)$ be the orbit through (S_0,n_0). Then $\Gamma^+(S_0,n_0)$ is bounded and precompact. (The proof of this result is rather technical and we shall omit it.) Therefore the omega-limit set $\Omega(S_0,n_0)$ is non-empty, compact and invariant and, moreover, $(S(t), n(t))\to\Omega(S_0,n_0)$ as $t\to\infty$. It can be shown that $\Omega(S_0,n_0)\subseteq C$. Now let $(\Sigma,\nu)\in\Omega(S_0,n_0)$, then it must be that $\Sigma = \hat{S}$ and $W[\nu] = \hat{W}$. Note that the function $\mathcal{V}$ of Exercise 5.6 also defines a Lyapunov function for the original system (5.1) - (5.3). The invariance of $\Omega(S_0,n_0)$ yields that for all $t\geqslant 0$ there exists an element $(\Sigma^{-t},\nu^{-t})\in\Omega(S_0,n_0)$ such that $T(t)(\Sigma^{-t},\nu^{-t}) = (\Sigma,\nu)$. Since $S(s;\Sigma^{-t},\nu^{-t}) = \hat{S}$ for all $s\geqslant 0$ we have that $n(s;\Sigma^{-t},\nu^{-t}) = T_D(s)\nu^{-t}$. In words: for initial pairs belonging to the omega-limit set the nonlinear problem reduces to the linear problem with $\beta(S(t))$ replaced by $\beta(\hat{S}) = D$. From theorem 5.4 we conclude that

$$\|T_D(s)\nu^{-t}-\hat{n}\| \leqslant Me^{-\epsilon s}\,\|\nu^{-t}\| \leqslant M'e^{-\epsilon s},\ s\geqslant 0\,,$$

where M' can be chosen independent of ν^{-t}, since $\Omega(S_0,n_0)$ is precompact. Substituting $s = t$ and using that $T_D(t)\nu^{-t} = \nu$ yields

$$\|\nu-\hat{n}\| \leqslant M'e^{-\epsilon t}\,,$$

and from the fact that this estimate is valid for all $t\geqslant 0$ we conclude that $\nu = \hat{n}$. Thus we have shown that

$$\Omega(S_0,n_0) = (\hat{S},\hat{n})\,,$$

and this proves the result.

EXERCISE 5.9. Let $D\geqslant D_{crit}$. Show that $\lim_{t\to\infty} (S(t),n(t)) = (S^i,0)$.

Hint: $\|n(t)\| = \int_{x_{\min}}^{1} n(t,x)dx \leqslant \frac{1}{x_{\min}} \int_{x_{\min}}^{1} xn(t,x)dx$.

Observe that contrary to the so-called "Stochastic Threshold" model discussed in section I.4.5, a change of the dilution rate D (which is a control parameter) does cause a deformation of the shape of the stable size distribution $\hat{n}$.

6. Interaction through the environment: some open problems.

The Daphnia model from section I.3 and variants of the cell proliferation models from the preceeding section (see the remarks in subsection 4.7 and remark 6.5 below) have in common that the individuals interact with each other only indirectly: they all consume from a common resource pool. A similar situation arises if the individuals produce a toxic chemical substance which has a restraining effect on their growth or an inhibiting effect on mitosis.

So consider the system of differential equations

$$\frac{dn}{dt} = A(S)n \tag{6.1}$$

$$\frac{dS}{dt} = F(S,L[n])\,, \tag{6.2}$$

where n takes values in an infinite-dimensional space X and S in $\mathbb{R}$, and where for each positive S the unbounded operator $A(S)$ generates a strongly continuous semigroup of bounded *linear* operators on X, which leaves a cone X_+ invariant. L denotes a continuous linear functional on X and F a smooth function of two real variables with $F(0,y)\geqslant 0$ for all $y\geqslant 0$. The idea is that the dynamical system generated by (6.1) - (6.2) should leave the cone $X_+\times\mathbb{R}_+$ invariant.

REMARK 6.1. Useful generalizations are immediate: i) S may take values in $\mathbb{R}^m$; ii) $\frac{dS}{dt} = F(S,L(S)[n])$, where for each $S\in\mathbb{R}^m$, $L(S)$ is a continuous linear function from X into $\mathbb{R}^k$ and where the mapping $S\mapsto L(S)$ from $\mathbb{R}^m$ into

$\mathcal{L}(X;\mathbb{R}^k)$ as well as $F:\mathbb{R}^m \times \mathbb{R}^k \rightarrow \mathbb{R}^m$ are smooth. But we don't want to complicate the formulation with technical details. Instead we present the simplest example which displays the essential features.

PROBLEM 6.2. Prove that the initial value problem for (6.1) - (6.2) is well-posed. That is, prove the existence and uniqueness of a solution for given initial data

$$n(0) = n_0 \in X_+ , \; S(0) = S_0 \geqslant 0$$

and prove that the solution depends continuously on (n_0,S_0), uniformly for t in compact sets. Here it seems crucial to formulate suitable hypotheses about the dependence of A on S. The *unbounded* generator A should depend smoothly on S, but in what sense? Actually, we may only need that the corresponding semigroups of *bounded* operators depend smoothly on S. So the Trotter-Kato ideas (PAZY, 1983, III.4) suggest to make assumptions about the dependence of the resolvent of A on S. Note that when S is a *given* function of time we may consider (6.1) as a non-autonomous *linear* evolution equation, and therefore the existing theory for these (PAZY, 1983, V) may prove useful. The crux of the matter is to find hypotheses which can be *easily* checked in concrete applications.

Let us assume that for fixed S the semigroup generated by $A(S)$ has a strictly dominant real eigenvalue $\lambda_d(S)$. So if $(\hat{n},\hat{S})$ is a nontrivial steady state then necessarily

$$\lambda_d(\hat{S}) = 0 \,, \tag{6.3}$$

and

$$\hat{n} = kn_d(\hat{S}) \,, \tag{6.4}$$

where $n_d(\hat{S})$ is the positive eigenvector corresponding to $\lambda_d(\hat{S})$, normalized in some convenient way. In many applications $\lambda_d(S)$ is a strictly monotone continuous function of S and in that case (6.3) has a unique solution $\hat{S}$ if $\lambda_d(0)<0<\lambda_d(\infty)$. Subsequently k has to be determined from the scalar equation

$$F(\hat{S},kL[n_d(\hat{S})]) = 0 \,. \tag{6.5}$$

One finds a unique solution if F is monotone in its second argument, which it usually is. Note that biological relevance requires that $k \geqslant 0$ (cf. I.3.5).

In order to linearize formally around an equilibrium $(\hat{n},\hat{S})$ we put

$$n = \hat{n} + u \quad , \quad S = \hat{S} + \xi \,,$$

and obtain, upon neglecting higher order terms,

$$\frac{du}{dt} = A(\hat{S})u + \xi A'(\hat{S})\hat{n} \tag{6.6}$$

$$\frac{d\xi}{dt} = \frac{\partial F}{\partial S}\xi + \frac{\partial F}{\partial y}L[u] \,. \tag{6.7}$$

where both partial derivatives of F are evaluated at $(\hat{S},L[\hat{n}])$. (Here we deliberately avoid to specify the sense in which the derivative $A'(\hat{S})$ is taken; so everything is purely formal and part of the problem will be to make this precise). For $\lambda \notin \sigma(A(\hat{S}))$ we can solve the first equation of the eigenvalue problem

$$(\lambda I - A(\hat{S}))u = \xi A'(\hat{S})\hat{n} \tag{6.8}$$

$$(\lambda - \frac{\partial F}{\partial S})\,\xi = \frac{\partial F}{\partial y}\,L[u] \,, \tag{6.9}$$

to obtain

$$u = \xi(\lambda I - A(\hat{S}))^{-1}A'(\hat{S})\hat{n} \,. \tag{6.10}$$

Upon substitution of this expression into (6.9) we find the *characteristic equation*

$$\lambda - \frac{\partial F}{\partial S} = \frac{\partial F}{\partial y}L((\lambda I - A(\hat{S}))^{-1}A'(\hat{S})\hat{n}) \,. \tag{6.11}$$

Note that $\lambda \in \sigma(A(\hat{S}))$ cannot belong to the point spectrum corresponding to (6.8) - (6.9) if $A'(\hat{S})\hat{n} \notin R(\lambda I - A(\hat{S}))$.

PROBLEM 6.3. Prove the Principle of Linearized Stability. That is, show that the stability of an equilibrium $(\hat{n},\hat{S})$ is determined by the position of the roots of (6.11) in the complex plane (giving due attention to the exceptional points $\lambda \in \sigma(A(\hat{S}))$).

Note that in general (6.1) - (6.2) is not semi-linear (i.e., it is not necessarily a bounded perturbation of a fixed unbounded generator) and hence the standard theory does not apply (in particular there is no immediate meaningful variation of constants formula). It is to be expected that the techniques which have to be developed for solving Problem 6.3 are useful as well (if not sufficient) for proving the Hopf bifurcation theorem for (6.1) - (6.2).

Of course it is a difficult problem in itself to determine whether all roots of (6.11) lie in the left half plane, or whether at least one root lies in the right half plane. Hence it is very important to develop analytical and numerical methods to decide about this question. Known results and approaches from the theory of delay equations (BELLMAN & COOKE 1963, COOKE 1985) could be taken as a starting point.

As argued in the preceding sections one can sometimes use monotonicity arguments to derive strong conclusions. Thus it is natural and important to pay attention to

PROBLEM 6.4. Find conditions on A, F and L which guarantee that the nonlinear semigroup corresponding to (6.1) - (6.2) is monotone (perhaps after restriction to a closed invariant subset).

Finally it would be advantageous to have a (partial) description of the special class of equations for which one can construct a Lyapunov function.

REMARK 6.5. Sometimes time scale arguments are used to replace (6.2) by the quasi-steady state equation

$$F(S, L[n]) = 0 .$$

If one can solve from this for S as a function of $L[n]$, substitution of the result into (6.1) leads to an S-independent problem. The cell proliferation model of section 4 is of this type. For this class of equations one can pose the same problems as above.

In the above we have concentrated on interaction through the environment which excludes (among other things) predator-prey and parasitoid-host interactions. Inclusion of such interactions leads to more complicated nonlinearities. For the corresponding class of equations (which is more difficult to describe abstractly in full generality) one has many open problems analogous to those described above. For age-dependent problems a reasonably complete mathematical theory is coming into existence. WEBB (1985a), PRÜSS (1983b) and SCHAPPACHER (in preparation) managed to prove the Principle of Linearized Stability, despite the fact that the problem is not semi-linear, but their proofs are still rather complicated and it will be very nice if a simple proof could be given. CUSHING (1983a) and PRÜSS (1983a) treat the Hopf bifurcation theorem (also see DIEKMANN & VAN GILS (1984) for those problems which can be reduced to Volterra convolution integral equations).

Notwithstanding that a lot of interesting and difficult mathematical problems arising in the context of structured populations have been solved, we can safely end this chapter, and in fact Part A of these lecture notes, by concluding that the field of *nonlinear* structured population models is largely a collection of open mathematical problems.

Note added in proof:
Very recently, Ph. CLÉMENT, O. DIEKMANN, M. GYLLENBERG, H.J.A.M. HEIJMANS and H.R. THIEME developed a perturbation theory for dual semigroups which yields a natural and very convenient framework for the local stability and bifurcation theory.

Bibliography

* : early papers which we found particularly stimulating.
m : mathematical survey paper or book.
b : biological survey paper or book.
mb : survey paper or book on mathematical biology.
s : paper dealing with one or more structured population models.
sm : s-paper with the stress on the mathematics.
sb : s-paper with the stress on the biology.

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